Problem 11840
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Proposed by G. Stoica (Canada).

Let $z_1, \ldots, z_n$ be complex numbers. Prove that

$$\left(\sum_{k=1}^{n} |z_k|\right)^2 - \left|\sum_{k=1}^{n} z_k\right|^2 \geq \left(\sum_{k=1}^{n} |Re(z_k)| - \left|\sum_{k=1}^{n} Re(z_k)\right|\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $z_k = x_k + iy_k$ for $k = 1, \ldots, n$, then, by Cauchy-Schwarz inequality,

$$|z_j||z_k| = \sqrt{|x_j|^2 + |y_j|^2} \cdot \sqrt{|x_k|^2 + |y_k|^2} \geq |x_j||x_k| + |y_j||y_k| \geq |x_j||x_k| + y_j y_k.$$

Hence

$$\begin{aligned}
\left(\sum_{k=1}^{n} |z_k|\right)^2 - \left|\sum_{k=1}^{n} z_k\right|^2 &= 2 \sum_{1 \leq j < k \leq n} |z_j||z_k| - 2 \sum_{1 \leq j < k \leq n} \mathrm{Re}(\overline{z_j} z_k) \\
&= 2 \sum_{1 \leq j < k \leq n} (|z_j||z_k| - (x_j x_k + y_j y_k)) \\
&\geq 2 \sum_{1 \leq j < k \leq n} (|x_j||x_k| - x_j x_k) \\
&= \left(\sum_{k=1}^{n} |x_k|\right)^2 - \left|\sum_{k=1}^{n} x_k\right|^2 \\
&= \left(\sum_{k=1}^{n} |x_k| + \left|\sum_{k=1}^{n} x_k\right|\right) \left(\sum_{k=1}^{n} |x_k| - \left|\sum_{k=1}^{n} x_k\right|\right) \\
&\geq \left(\sum_{k=1}^{n} |x_k| - \left|\sum_{k=1}^{n} x_k\right|\right)^2 = \left(\sum_{k=1}^{n} |\mathrm{Re}(z_k)| - \left|\sum_{k=1}^{n} \mathrm{Re}(z_k)\right|\right)^2,
\end{aligned}$$

where we also applied the inequality $\sum_{k=1}^{n} |x_k| \geq |\sum_{k=1}^{n} x_k|$. □