Problem 11837
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Let $a_0 = 1$, *and for* $n \geq 0$ *let* $a_{n+1} = a_n + e^{-a_n}$. *Let* $b_n = a_n - \log n$. *For* $n \geq 0$, *show that* $0 < b_{n+1} < b_n$ *and also show that* $\lim_{n\to\infty} b_n = 0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first show by induction that $a_n \geq \log(n+1)$. It holds for $n = 0$ and for $n \geq 0$,

$$a_{n+1} = f(a_n) \geq f(\log(n+1)) = \log(n+1) + \frac{1}{n+1} \geq \log(n+1) + \log\left(1 + \frac{1}{n+1}\right) = \log(n+2)$$

where $f(x) = x + e^{-x}$ is an increasing function for $x \geq 0$.
Hence $b_n \geq \log(n+1) - \log(n) > 0$, and

$$b_{n+1} - b_n = e^{-a_n} + \log\left(1 - \frac{1}{n+1}\right) \leq -\left(-\frac{1}{n+1} - \log\left(1 - \frac{1}{n+1}\right)\right) < 0$$

Since $\{b_n\}_{n\geq 0}$ is strictly decreasing positive sequence, it follows that it has a finite limit $L \in [0, 1)$. Note that

$$(n+1)(b_{n+1} - b_n) = (n+1)\left(\frac{e^{-b_n}}{n} + \log\left(1 - \frac{1}{n+1}\right)\right) \to e^{-L} - 1.$$

Finally, let $H_n = \sum_{k=1}^{n} 1/k$, then, by applying Stolz-Cesaro theorem, we have

$$0 = \frac{L}{+\infty} = \lim_{n\to\infty} \frac{b_n}{H_n} \stackrel{\text{SC}}{=} \lim_{n\to\infty} \frac{b_{n+1} - b_n}{1/(n+1)} = e^{-L} - 1.$$

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