Problem 11836
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Proposed by T. Viteam (Uruguay).

Let ABC be a triangle with sides of lengths a, b, and c, circumradius R, and inradius r. For $x, y, z > 0$, let

$$f(x,y,z) = \frac{xyz}{(x+y)(z^2-(x-y)^2)}.$$

Prove that

$$\frac{3R}{4r} \geq f(a,b,c) + f(b,c,a) + f(c,a,b).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that

$$r^2 = \frac{(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)} \qquad \text{and} \qquad R = \frac{abc}{2r(a+b+c)},$$

which imply that

$$\frac{3R}{4r} = \frac{3abc}{2(b+c-a)(a+c-b)(a+b-c)}.$$

Moreover

$$f(a,b,c) + f(b,c,a) + f(c,a,b) = \frac{abc}{(a+b)(b+c-a)(a+c-b)} + \frac{abc}{(b+c)(a+c-b)(a+b-c)} + \frac{abc}{(a+c)(b+c-a)(a+b-c)}.$$

Therefore, the required inequality is equivalent to

$$\frac{3}{2} \geq \frac{a+b-c}{a+b} + \frac{b+c-a}{b+c} + \frac{a+c-b}{a+c} = 3 - \left(\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{a+c}\right)$$

or

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{a+c} = h\left(\frac{c}{a+b+c}\right) + h\left(\frac{a}{a+b+c}\right) + h\left(\frac{b}{a+b+c}\right) \geq 3h(1/3) = \frac{3}{2}$$

which holds because

$$h(x) = \frac{x}{1-x}$$

is convex in $(0,1)$. ∎