Problem 11835
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Proposed by G. Stoica (Canada).

Find all functions f from $[0,\infty)$ to $[0,\infty)$ such that whenever $x, y \geq 0$,

$$\sqrt{3}f(2x) + 5f(2y) \leq 2f(\sqrt{3}x + 5y).$$

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

We will show a more general statement.
Let $a \in (0,1)$ and $b \in (1,+\infty)$ such that $a^m b^n \neq 1$ for all $n, m \in \mathbb{N}^+$.
If $f : [0,\infty) \to [0,\infty)$ is a function such that whenever $x, y \geq 0$, $af(x) + bf(y) \leq f(ax + by)$, then $f(t) = ct$ for some $c \in [0,+\infty)$.
The solution to the proposed problem is obtained when $a = \sqrt{3}/2$ and $b = 5/2$.
We divide the proof into several step.

i) The set $S := \{a^m b^n \ : \ m, n \in \mathbb{N}\}$ is dense in $[0,+\infty)$.
It follows from the fact that the set $\{\log_b(a^m b^n) \ : \ m, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$. In fact

$$\log_b(a^m b^n) = n - m\log_b(1/a) = n - \lfloor m\alpha \rfloor - \{m\alpha\},$$

where $\alpha = \log_b(1/a)$ is a positive irrational number (otherwise there would be $p, q \in \mathbb{N}^+$ such that $1/a = b^{p/q}$, that is $a^q b^p = 1$). It suffices to note that $\{\{m\alpha\} : m \in \mathbb{N}\}$ is dense in $[0,1]$.

ii) $f(0) = 0$.
Let $x = y = 0$, then $(a + b - 1)f(0) = af(0) + bf(0) - f(a0 + b0) \leq 0$ which implies that $f(0) = 0$ because $f(0) \geq 0$ and $a + b > b > 1$.

iii) For $s \in S$, and $t \geq 0$, $sf(t) \leq f(st)$.
By definition $s = a^m b^n$ for some $m, n \in \mathbb{N}$. If $m = n = 0$ then $s = 1$ and the inequality is trivial. Moreover, by the inductive assumption,

$$a^m b^{n+1} f(t) \leq bf(a^m b^n t) \leq af(0) + bf(a^m b^n t) \leq f(a0 + b(a^m b^n t)) = f(a^m b^{n+1} t).$$

In a similar way,

$$a^{m+1} b^n f(t) \leq af(a^m b^n t) \leq af(a^m b^n t) + bf(0) \leq f(a(a^m b^n t) + b0) = f(a^{m+1} b^n t).$$

iv) For all $t \geq 0$, $f(t) \leq f(1)t$.
By ii), it holds for $t = 0$. Let $t > 0$, then by i) there exists a sequence $\{s_k\}_{k\geq 0}$ in S such that $s_k \to (1/(bt))^-$. Hence, by iii),

$$f(t) \leq \frac{f(s_k t)}{s_k} = \frac{bf(s_k t)}{bs_k} \leq \frac{af(x_k) + bf(s_k t)}{bs_k} \leq \frac{f(ax_k + bs_k t)}{bs_k} = \frac{f(1)}{bs_k}$$

where $x_k = (1 - bs_k t)/a > 0$. The inequality follows by taking the limit as k goes to infinity.

v) For all $t \geq 0$, $f(t) = f(1)t$.
By ii), it holds for $t = 0$. Let $t > 0$, then also the function $h_t(x) := f(tx)$ satisfies the hypothesis. Hence, by iv), for all $x \geq 0$, $h_t(x) \leq h_t(1)x$. Finally, by letting $x = 1/t > 0$, together with iv), we obtain

$$f(t) \leq f(1)t = h_t(1/t)t \leq h_t(1)(1/t)t = f(t).$$

□