Problem 11833
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Proposed by M. Safaryan (Armenia) and V. Aslanyan (UK).

Let f be a real-valued function on an open interval (a,b) such that the one-sided limits $\lim_{t\to x^-} f(t)$ and $\lim_{t\to x^+} f(t)$ exist and are finite for all x in (a,b). Can the set of discontinuities of f be uncountable?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The answer is no. For all $n \in \mathbb{N}^+$, let

$$D_n = \{x \in (a,b) : \forall r > 0,\ \exists y \in (x-r, x+r) \cap (a,b)\ :\ |f(x) - f(y)| > 1/n\}.$$

Hence the set of all discontinuities of f is $\bigcup_{n>0} D_n$.
Assume by contradiction that it is uncountable, then there is a positive integer n such that D_n is uncountable. Therefore D_n has at least an accumulation point z, which means that

$$\forall k \in \mathbb{N}^+,\ ((z-1/k, z) \cup (z, z+1/k)) \cap D_n \neq \emptyset.$$

Therefore

$$\text{i) } \forall k \in \mathbb{N}^+,\ (z-1/k, z) \cap D_n \neq \emptyset \qquad \text{or} \qquad \text{ii) } \forall k \in \mathbb{N}^+,\ (z, z+1/k) \cap D_n \neq \emptyset.$$

Suppose that i) holds (the other case is similar). For $k \in \mathbb{N}^+$, let $x_k \in (z-1/k, z) \cap D_n$ and let $0 < r < 1/k$ such that $(x_k - r, x_k + r) \subset (z - 1/k, z)$.
Then, by the definition of D_n, there is $y_k \in (x_k - r, x_k + r) \cap (a,b)$ such that

$$|f(x_k) - f(y_k)| > 1/n. \tag{1}$$

On the other hand, $x_k \to z^-$, $y_k \to z^-$ and therefore, by hypothesis,

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f(y_k)$$

which is in contradiction with (1). $\square$