Problem 11832
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Proposed by D. Knuth (USA).

Let $C(z)$ be the generating function of the Catalan numbers. Prove that

$$(\log(C(z)))^2 = \sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n},$$

where $H_k = \sum_{j=1}^{k} 1/j$ is the k-th harmonic number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We note that

$$\begin{aligned}\sum_{n=1}^{\infty} \binom{2n}{n} \frac{z^n}{n} &= \int_0^z \sum_{n=1}^{\infty} \binom{2n}{n} x^{n-1} dx = \int_0^z \left(\frac{1}{\sqrt{1-4x}} - 1 \right) \frac{dx}{x} \\ &= \int_{1-4z}^{1} \left(\frac{1}{\sqrt{t}} - 1 \right) \frac{dt}{1-t} = \int_{1-4z}^{1} \frac{1}{\sqrt{t}(1+\sqrt{t})} dt \\ &= \Big[2\ln(1+\sqrt{t}) \Big]_{1-4z}^{1} = 2\log(C(z)).\end{aligned}$$

Hence

$$\begin{aligned}(\log(C(z)))^2 &= \frac{1}{4} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \left(\frac{1}{k(n-k)} \binom{2k}{k} \binom{2(n-k)}{n-k} \right) z^n \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \left(\left(\frac{1}{k} + \frac{1}{n-k} \right) \binom{2k}{k} \binom{2(n-k)}{n-k} \right) \frac{z^n}{n} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \left(\frac{1}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \right) \frac{z^n}{n}.\end{aligned}$$

Therefore, it suffices to show by induction that

$$\sum_{k=1}^{n-1} F(n,k) = 2(H_{2n-1} - H_n) \quad \text{where} \quad F(n,k) = \frac{1}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{2n}{n}^{-1}.$$

It holds for $n = 1$, and it is easy to verify that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k) \quad \text{where} \quad G(n,k) = -\frac{k^2(2n-2k+1)F(n,k)}{(n+1)(2n+1)(n+1-k)}.$$

Hence, by the inductive assumption,

$$\begin{aligned}\sum_{k=1}^{n} F(n+1,k) &= \sum_{k=1}^{n} F(n,k) + \sum_{k=1}^{n} (G(n,k+1) - G(n,k)) \\ &= 2(H_{2n-1} - H_n) + F(n,n) + G(n,n+1) - G(n,1) \\ &= 2(H_{2n-1} - H_n) + \frac{1}{n} + 0 + \frac{(2n-1)F(n,1)}{(n+1)(2n+1)n} \\ &= 2(H_{2n-1} - H_n) + \frac{1}{n} + \frac{1}{(n+1)(2n+1)} = 2(H_{2n+1} - H_{n+1}).\end{aligned}$$

□