Problem 11831
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Proposed by R. Ozols (Latvia).

Prove that for $\varepsilon > 0$ there exists an integer n such that the greatest prime divisor of n^2+1 is less than εn.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For all $k \in \mathbb{N}^+$, let $n_k = (k-1)(k^4+1)+k^2$, then n_k is a positive integer such that n_k^2+1 can be written as a product of three integer factors

$$n_k^2+1 = (k^2-2k+2)\cdot(k^4-k^2+1)\cdot(k^4+1).$$

Thus $P(n_k^2+1)$, the greatest prime divisor of n_k^2+1, is less or equal to

$$\max((k^2-2k+2),(k^4-k^2+1),(k^4+1)) = k^4+1$$

and

$$0 < \frac{P(n_k^2+1)}{n_k} \leq \frac{k^4+1}{(k-1)(k^4+1)+k^2}$$

which implies that, as $k \to \infty$,

$$\frac{P(n_k^2+1)}{n_k} \to 0^+.$$

□

The above statement says that

$$\liminf_{n\to\infty} \frac{P(n^2+1)}{n} = 0.$$

On the other hand, it has been proved by Chebyshev that

$$\limsup_{n\to\infty} \frac{P(n^2+1)}{n} = +\infty.$$

Moreover it is easy to show that

$$\lim_{n\to\infty} P(n^2+1) = +\infty$$

(it is a well-known conjecture that there are infinitely many primes of the form n^2+1).
Suppose that for infinitely many n the prime factors of n^2+1 are included in the finite set $\{p_1, p_2, \dots, p_r\}$. Then infinitely many integers n^2+1 can be written in the form dy^3 where d belongs to the finite set

$$D := \{p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \ : \ a_i \in \{0,1,2\} \text{ for } i = 1,2,\dots,r\}.$$

By letting $x = n$, it follows that for at least one of those d, there are infinitely many solutions to the diophantine equation

$$x^2 - dy^3 = -1,$$

which is a contradiction because, by Siegel's theorem, it has finitely many solutions.