

Problem 11829

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Proposed by P. Bracken (USA).

Let $\{a_n\}_{n \geq 1}$ be a monotone decreasing sequence of real numbers that converges to 0. Prove that $\sum_{n=1}^{\infty} a_n/n < \infty$ if and only if $a_n = O(1/\log(n))$ and $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \log(n) < \infty$.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

Let

$$S_N = \sum_{n=1}^N \frac{a_n}{n} \quad \text{and} \quad T_N = \sum_{n=1}^N (a_n - a_{n+1}) \log(n).$$

If $\{a_n\}_{n \geq 1}$ is a monotone and decreasing to zero then $\{S_N\}_{N \geq 1}$ and $\{T_N\}_{N \geq 1}$ are increasing. Let S and T their limits. Notice that for $n \geq 2$,

$$\log(n) - \log(n-1) = \int_{n-1}^n \frac{dx}{x} \in \left(\frac{1}{n}, \frac{1}{n-1} \right).$$

i) If $S < \infty$ then for $N \geq 2$,

$$a_N \log(N) \leq a_N \sum_{n=2}^N (\log(n) - \log(n-1)) \leq a_N \sum_{n=2}^N \frac{1}{n-1} \leq \sum_{n=2}^N \frac{a_{n-1}}{n-1} = S_{N-1} \leq S$$

which implies that $a_n = O(1/\log(n))$. Moreover

$$\begin{aligned} T_N &= \sum_{n=1}^N (a_n - a_{n+1}) \log(n) = \sum_{n=2}^N a_n \log(n) - \sum_{n=2}^{N+1} a_n \log(n-1) \\ &= \sum_{n=2}^N a_n (\log(n) - \log(n-1)) - a_N \log(N-1) \\ &\leq \sum_{n=2}^N \frac{a_n}{n-1} \leq \sum_{n=2}^N \frac{a_{n-1}}{n-1} = S_{N-1} \leq S \end{aligned}$$

which implies that $T < \infty$.

ii) If $T < \infty$ and $a_N \log(N) \leq M$ for $N \geq 2$, then

$$\begin{aligned} S_N - a_1 &= \sum_{n=2}^N \frac{a_n}{n} \leq \sum_{n=2}^N a_n (\log(n) - \log(n-1)) = \sum_{n=2}^N a_n \log(n) - \sum_{n=2}^{N-1} a_{n+1} \log(n) \\ &= \sum_{n=2}^{N-1} (a_n - a_{n+1}) \log(n) + a_N \log(N) = T_{N-1} + a_N \log(N) \leq T + M, \end{aligned}$$

which implies that $S < \infty$.