Problem 11828
(American Mathematical Monthly, Vol.122, March 2015)

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Let n be a positive integer, and let z be a complex number that is not a kth root of unity for any k with $1 \leq k \leq n$. Let S be the set of all lists $(a_1, \ldots, a_n)$ of n nonnegative integers such that $\sum_{k=1}^{n} k a_k = n$. Prove that

$$\sum_{a \in S} \prod_{k=1}^{n} \frac{1}{a_k! k^{a_k} (1-z^k)^{a_k}} = \prod_{k=1}^{n} \frac{1}{1-z^k}.$$

For example, for $n = 3$ we have

$$\frac{1}{6(1-z)^3} + \frac{1}{2(1-z)(1-z^2)} + \frac{1}{3(1-z^3)} = \frac{1}{(1-z)(1-z^2)(1-z^3)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First solution. We have that

$$\begin{aligned}
F(z,t) &:= \sum_{n\geq 0} t^n \sum_{\alpha_1+2\alpha_2+\cdots=n} \prod_{k=1}^{n} \frac{1}{\alpha_k! k^{\alpha_k} (1-z^k)^{\alpha_k}} \\
&= \prod_{k=1}^{n} \sum_{\alpha_k \geq 0} \frac{t^{k\alpha_k}}{\alpha_k! k^{\alpha_k} (1-z^k)^{\alpha_k}} \\
&= \prod_{k\geq 1} \exp\left(\frac{t^k}{k(1-z^k)}\right) \\
&= \exp\left(\sum_{k\geq 1} \frac{t^k}{k(1-z^k)}\right) \\
&= \exp\left(\sum_{j\geq 0}\sum_{k\geq 1} \frac{(tz^j)^k}{k}\right) \\
&= \prod_{j\geq 0} \exp\left(\sum_{k\geq 1} \frac{(tz^j)^k}{k}\right) \\
&= \prod_{j\geq 0} \exp\left(\ln\left(\frac{1}{1-tz^j}\right)\right) \\
&= \prod_{j\geq 0} \frac{1}{1-tz^j} = \frac{1}{1-t}\prod_{j\geq 1} \frac{1}{1-tz^j}
\end{aligned}$$

Note that $[t^n z^N]F(z,t)$ is the number of integer partitions of N with at most n parts because it is known that

$$[t^n z^N] \prod_{j\geq 1} \frac{1}{1-tz^j}$$

is the number of integer partitions of N with exactly n parts. On the other side,

$$G(z,t) := \sum_{n\geq 0} t^n \prod_{k=1}^{n} \frac{1}{1-z^k}$$

and $[t^n z^N]G(z,t)$ is the number of integer partitions of N with at most n parts.
Hence F and G are equal and the identity is proved. $\square$

Second solution. It suffices to prove that

$$\sum_{\sigma\in\pi(I_n)}\prod_{k=1}^{n}\frac{1}{1-\prod_{j=1}^{k}x_{\sigma(j)}}=\sum_{\sigma\in\pi(I_n)}\prod_{k=1}^{l_\sigma}\frac{1}{1-\prod_{j\in C_k}x_j}$$

where $\sigma\in\pi(I_n)$ is a permutation of $I_n=\{1,2,\ldots,n\}$ which can be written as a product of disjoint cycles $C_1,C_2,\ldots,C_{l_\sigma}$. The required identity follows by letting $x_j=z$ for $j=1,\ldots,n$.
We note that

$$\prod_{k=1}^{n}\frac{1}{1-\prod_{j=1}^{k}x_{\sigma(j)}}=\sum_{a_1\geq a_2\geq\cdots\geq a_n\geq 0}\prod_{j=1}^{n}x_{\sigma(j)}^{a_j},$$

and

$$\prod_{k=1}^{l_\sigma}\frac{1}{1-\prod_{j\in C_k}x_j}=\sum_{a_1\geq 0,a_2\geq 0,\cdots,a_{l_\sigma}\geq 0}\prod_{k=1}^{l_\sigma}\left(\prod_{j\in C_k}x_j\right)^{a_k}.$$

Hence, if $a_1=a_2=\cdots=a_n$ then

$$[x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}]\sum_{\sigma\in\pi(I_n)}\prod_{k=1}^{n}\frac{1}{1-\prod_{j=1}^{k}x_{\sigma(j)}}=n!=[x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}]\sum_{\sigma\in\pi(I_n)}\prod_{k=1}^{l_\sigma}\frac{1}{1-\prod_{j\in C_k}x_j}.$$

In general,

$$[x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}]\sum_{\sigma\in\pi(I_n)}\prod_{k=1}^{n}\frac{1}{1-\prod_{j=1}^{k}x_{\sigma(j)}}=\prod_{i\geq 0}(|A_i|!)$$

where $A_i=\{j: a_j=i\}$ for $i\geq 0$. By the above remark,

$$|A_i|!=[\prod_{j\in A_i}x_j^{a_j}]\sum_{\sigma_i\in\pi(A_i)}\prod_{k=1}^{l_{\sigma_i}}\frac{1}{1-\prod_{j\in C_{k,i}}x_j}.$$

Therefore

$$\begin{aligned}\prod_{i\geq 0}(|A_i|!)&=\prod_{i\geq 0}[\prod_{j\in A_i}x_j^{a_j}]\sum_{\sigma_i\in\pi(A_i)}\prod_{k=1}^{l_{\sigma_i}}\frac{1}{1-\prod_{j\in C_{k,i}}x_j}\\&=[x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}]\sum_{\sigma\in\pi(I_n)}\prod_{k=1}^{l_\sigma}\frac{1}{1-\prod_{j\in C_k}x_j}\end{aligned}$$

where σ is the product of $\sigma_i\in\pi(A_i)$ for $i\geq 0$. $\square$