Problem 11827
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Proposed by G. Stoica (Canada).

Show that there are infinitely many rational triples (a, b, c) such that $a + b + c = abc = 6$.

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Let $x = a$ and $y = b$ then $a + b + c = abc = 6$ is equivalent to

$$F(x, y) := xy(6 - x - y) - 6 = 0$$

which is a cubic plane algebraic curve. Since

$$\frac{\partial F}{\partial x}(x, y) = y(6 - 2x - y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = x(6 - 2y - x),$$

it follows that its critical points are $(0, 0)$, $(6, 0)$, $(0, 6)$, and $(2, 2)$. None of them belongs to the curve, therefore it is non-singular and the homogeneous equation

$$xy(6z - x - y) - 6z^3 = 0$$

is an elliptic curve in the projective plane. By Mordell's Theorem, the points on an elliptic curve with rational coordinates form a finitely generated abelian group. Moreover, by Mazur's Theorem, an elliptic curve has at most 16 rational torsion points (i. e. points of finite order). This means that if an elliptic curve has at least 17 distinct rational points, then it has infinitely many. In fact our curve satisfies this condition: we have 3 rational points *at infinity* $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$ and at least 18 *real* rational points

$$\begin{array}{cccccc}
(1, 2, 1), & (2, 1, 1), & (1, 3, 1), & (3, 1, 1), & (2, 3, 1), & (3, 2, 1), \\
(-\frac{1}{2}, -\frac{3}{2}, 1), & (-\frac{3}{2}, -\frac{1}{2}, 1), & (-\frac{1}{2}, 8, 1), & (8, -\frac{1}{2}, 1), & (-\frac{3}{2}, 8, 1), & (8, -\frac{3}{2}, 1), \\
(\frac{49}{15}, \frac{25}{21}, 1), & (\frac{25}{21}, \frac{49}{15}, 1), & (\frac{25}{21}, \frac{54}{35}, 1), & (\frac{54}{35}, \frac{25}{21}, 1), & (\frac{49}{15}, \frac{54}{35}, 1), & (\frac{54}{35}, \frac{49}{15}, 1).
\end{array}$$

$\square$