Problem 11826
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Proposed by M. Bataille (France).

Let m and n be positive integers with $m \leq n$. Prove that

$$\sum_{k=m}^{n} 4^{n+1-k}\binom{m+k-1}{m-1}^2 \geq \sum_{k=m}^{n}\binom{m+n}{k}^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $1 \leq m \leq n$, let

$$F_m(n) = \sum_{k=m}^{n} \frac{1}{4^k}\binom{m+k-1}{m-1}^2 - \frac{1}{4^{n+1}}\sum_{k=m}^{n}\binom{m+n}{k}^2.$$

Then

$$F_m(m) = \frac{1}{4^m}\binom{2m-1}{m-1}^2 - \frac{1}{4^{m+1}}\binom{2m}{m}^2 = 0.$$

So it suffices to show that $F_m(n) \geq F_m(n-1)$ for $n > m$. Note that

$$\sum_{k=m}^{n}\binom{m+n}{k}^2\frac{k}{m+n} = \sum_{k=m}^{n}\binom{m+n}{m+n-k}^2\frac{m+n-k}{m+n} = \sum_{k=m}^{n}\binom{m+n}{k}^2 - \sum_{k=m}^{n}\binom{m+n}{k}^2\frac{k}{m+n}$$

implies that

$$\sum_{k=m}^{n}\binom{m+n}{k}^2\frac{2k}{m+n} = \sum_{k=m}^{n}\binom{m+n}{k}^2.$$

Hence

$$\begin{aligned}
F_m(n) - F_m(n-1) &= \frac{1}{4^n}\binom{m+n-1}{m-1}^2 - \frac{1}{4^{n+1}}\sum_{k=m}^{n}\binom{m+n}{k}^2 + \frac{1}{4^n}\sum_{k=m}^{n-1}\binom{m+n-1}{k}^2 \\
&= \frac{1}{4^n}\sum_{k=m-1}^{n-1}\binom{m+n-1}{k}^2 - \frac{1}{4^{n+1}}\sum_{k=m}^{n}\binom{m+n}{k}^2 \\
&= \frac{1}{4^{n+1}}\left(\sum_{k=m}^{n}\binom{m+n}{k}^2\left(\frac{2k}{m+n}\right)^2 - \sum_{k=m}^{n}\binom{m+n}{k}^2\right) \geq 0,
\end{aligned}$$

because by Cauchy-Schwarz inequality

$$\sum_{k=m}^{n}\binom{m+n}{k}^2 \cdot \sum_{k=m}^{n}\binom{m+n}{k}^2\left(\frac{2k}{m+n}\right)^2 \geq \left(\sum_{k=m}^{n}\binom{m+n}{k}^2\frac{2k}{m+n}\right)^2 = \left(\sum_{k=m}^{n}\binom{m+n}{k}^2\right)^2.$$

□