Problem 11825
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Proposed by M. Dincă and S. Radulescu (Romania).

Let E be a normed linear space. Given $x_1, \dots x_n \in E$ (with $n \geq 2$) such that $\|x_k\| = 1$ for $1 \leq k \leq n$ and the origin of E is in the convex hull of $\{x_1, \dots, x_n\}$, prove that $\|x_1 + \cdots + x_n\| \leq n - 2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since the origin of E is in the convex hull of $\{x_1, \dots, x_n\}$, it follows that there exist non-negative numbers $\alpha_1, \dots, \alpha_n$ such that

$$\alpha_1 + \cdots + \alpha_n = 1 \qquad \text{and} \qquad \alpha_1 x_1 + \cdots + \alpha_n x_n = 0.$$

We show that $\alpha_k \in [0, 1/2]$ for $1 \leq k \leq n$. In fact

$$\alpha_1 = \| - \alpha_1 x_1\| = \|\alpha_2 x_2 + \cdots + \alpha_n x_n\| \leq \alpha_2 \|x_2\| + \cdots + \alpha_n \|x_n\| = \alpha_2 + \cdots + \alpha_n = 1 - \alpha_1.$$

which implies that $\alpha_1 \leq 1/2$. The other cases are similar. Hence

$$\begin{aligned}\|x_1 + \cdots + x_n\| &= \|x_1 + \cdots + x_n - 2(\alpha_1 x_1 + \cdots + \alpha_n x_n)\| = \|(1 - 2\alpha_1)x_1 + \cdots + (1 - 2\alpha_n)x_n\| \\ &\leq (1 - 2\alpha_1)\|x_1\| + \cdots + (1 - 2\alpha_n)\|x_n\| = (1 - 2\alpha_1) + \cdots + (1 - 2\alpha_n) = n - 2.\end{aligned}$$

Finally, we note that right-hand side of the inequality can not be lowered. Take any unit vector x_1, let $x_2 = \cdots = x_{n-1} = x_1$ and $x_n = -x_1$. Then the origin of E is in the convex hull of $\{x_1, \dots, x_n\}$ because $\frac{1}{2}x_1 + \frac{1}{2}x_n = 0$, and $\|x_1 + \cdots + x_n\| = \|(n - 2)x_1\| = n - 2$.