Problem 11822
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Proposed by G. Stoica (Canada).

Call a polynomial real if all its coefficients are real. Let P and Q be polynomials with complex coefficients such that the composition $P \circ Q$ is real. Show that if the leading coefficient of Q and its constant term are both real, then P and Q are real.

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Our proof is inspired by A. Horwitz's paper *Compositions of polynomials with coefficients in a given field*, J. Math. Anal. Appl. 267 (2002), no. 2, 489-500. The same proof still holds if we replace $\mathbb{R}$ and $\mathbb{C}$ with two fields of characteristic 0, F_1 and F_2, such that $F_1 \subset F_2$.
We assume that P and Q are not constant, otherwise the statement is false:

$$P(x) = 0, \quad Q(x) = x^2 + ix, \quad P(Q(x)) = 0 \quad \text{and} \quad P(x) = ix, \quad Q(x) = 0, \quad P(Q(x)) = 0.$$

Let $P, Q \in \mathbb{C}[x]$ such that $P \circ Q \in \mathbb{R}[x]$ with

$$P(x) = \sum_{k=0}^{n} a_k x^k, \quad Q(x) = \sum_{j=0}^{m} b_j x^j \quad \text{and} \quad P(Q(x)) = \sum_{i=0}^{mn} c_i x^i.$$

where $n, m \geq 1$, $b_m, b_0 \in \mathbb{R}$, and $a_n \neq 0$, $b_m \neq 0$. Note that $a_n = c_{mn}/b_m \in \mathbb{R}$.

i) We show that $Q \in \mathbb{R}[x]$.
Assume by contradiction that Q is not real, and let $d = \min\{j : b_{m-j} \notin \mathbb{R}\} \in [1, m-1]$.
Since $mn - d > m(n-1)$, it follows that

$$c_{mn-d} = a_n [x^{mn-d}] Q(x)^n = a_n \sum_{\substack{k_0 + \cdots k_m = n \\ 0k_0 + \cdots m k_m = mn-d}} \binom{n}{k_0, \dots k_m} (b_0)^{k_0} \cdots (b_m)^{k_m}.$$

Moreover

$$\sum_{i=0}^{m-1} (m-i) k_i = m \sum_{i=0}^{m} k_i - \sum_{i=0}^{m} i k_i = mn - (mn - d) = d,$$

which implies that $k_i = 0$ for $i \in [0, m-d-1)]$ (otherwise the above sum is greater than d).
Hence

$$d k_{m-d} + \sum_{i=m-d+1}^{m-1} (m-i) k_i = d$$

If $k_{m-d} > 0$ then $k_{m-d} = 1$ and $k_i = 0$ for $i \in [m-d+1, m-1]$. Therefore

$$c_{mn-d} = a_n n b_{m-d} b_m^{n-1} + a_n \sum \binom{n}{k_{m-d+1}, \dots k_m} (b_{m-d+1})^{k_{m-d+1}} \cdots (b_m)^{k_m}.$$

Since $a_n \neq 0$, $b_m \neq 0$, and $c_{mn-d}, a_n, b_{m-d+1}, \dots, b_m \in \mathbb{R}$, it follows that also $b_{m-d} \in \mathbb{R}$ which contradicts the definition of d.

ii) We show that $P \in \mathbb{R}[x]$.
Assume by contradiction that P is not real, and let $d = \min\{j : a_{n-j} \notin \mathbb{R}\} \in [1, n]$. Then

$$c_{mn-md} = [x^{m(n-d)}] \sum_{k=n-d+1}^{n} a_k Q(x)^k + a_{n-d} b_m^{n-d}.$$

Since $a_n \neq 0$, $b_m \neq 0$, $Q \in \mathbb{R}[x]$ and $c_{mn-md}, a_{n-d+1}, \dots, a_n \in \mathbb{R}$, it follows that also $a_{n-d} \in \mathbb{R}$ which contradicts the definition of d.

□