Problem 11821
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Proposed by F. Holland and C. Koester (Ireland).

Let p be a positive integer. Prove that

$$\lim_{n\to\infty}\frac{1}{2^n n^p}\sum_{k=0}^{n}(n-2k)^{2p}\binom{n}{k}=\prod_{j=1}^{p}(2j-1).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$\begin{aligned}\frac{1}{2^n}\sum_{k=0}^{n}(n-2k)^{2p}\binom{n}{k}&=\frac{1}{2^n}\sum_{j=0}^{2p}\binom{2p}{j}n^{2p-j}\cdot\sum_{k=0}^{n}\binom{n}{k}(-2k)^j\\&=\frac{1}{2^n}\sum_{j=0}^{2p}\binom{2p}{j}\left[\frac{x^{2p-j}}{(2p-j)!}\right]e^{nx}\cdot\left[\frac{x^j}{j!}\right](1+e^{-2x})^n\\&=\left[\frac{x^{2p}}{(2p)!}\right]\left(\frac{e^x+e^{-x}}{2}\right)^n=D^{2p}((\cosh(x))^n)(0).\end{aligned}$$

The Faá di Bruno's formula says that

$$D^{2p}(f\circ g)(x)=\sum\frac{(2p)!}{\prod_{j=1}^{2p}(k_j)!(j!)^{k_j}}\cdot(D^k f)(g(x))\cdot\prod_{j=1}^{2p}((D^j g)(x))^{k_j}$$

where $k=\sum_{j=1}^{2p}k_j$ and the sum is taken over all $2p$-tuples of nonnegative integers $(k_1,\dots,k_{2p})$ for which $\sum_{j=1}^{2p}jk_j=2p$.
In our case $f(x)=x^n$ and $g(x)=\cosh(x)$ and since

$$D^k(x^n)(\cosh(0))=\prod_{j=0}^{k-1}(n-j)\qquad\text{and}\qquad D^j(\cosh)(0)=\begin{cases}1 & \text{if } j \text{ is even}\\ 0 & \text{if } j \text{ is odd}\end{cases},$$

it follows that we can take $k_1=k_3=\cdots=k_{2p-1}=0$.
Hence $k=\sum_{j=1}^{p}k_{2j}$, the costraint becomes $\sum_{j=1}^{p}jk_{2j}=p$, and

$$D^{2p}((\cosh(x))^n)(0)=\sum\frac{(2p)!}{\prod_{j=1}^{p}(k_{2j})!((2j)!)^{k_{2j}}}\cdot\prod_{j=0}^{k-1}(n-j).$$

The RHS is a polynomial in n of degree the maximum value of k which is equal to p. It is attained only when $k_2=p$ and $k_4=\cdots=k_{2p}=0$, therefore we finally find that

$$\lim_{n\to\infty}\frac{1}{2^n n^p}\sum_{k=0}^{n}(n-2k)^{2p}\binom{n}{k}=\lim_{n\to\infty}\frac{D^{2p}((\cosh(x))^n)(0)}{n^p}=\frac{(2p)!}{p!(2!)^p}=\prod_{j=1}^{p}(2j-1).$$

□