Problem 11819
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Proposed by C. Lupu (USA).

Let f be a continuous, nonnegative function on $[0,1]$. *Show that*

$$\int_0^1 f^3(x)\,dx \geq 4\left(\int_0^1 x^2 f(x)\,dx\right)\left(\int_0^1 x f^2(x)\,dx\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show a more general result.
Let f and g be continuous, nonnegative functions on $[0,1]$ and let a and b be nonnegative real numbers. Then

$$\left(\int_0^1 f^{a+b}(x)\,dx\right)\left(\int_0^1 g^{a+b}(x)\,dx\right) \geq \left(\int_0^1 f^a(x)g^b(x)\,dx\right)\left(\int_0^1 f^b(x)g^a(x)\,dx\right).$$

The required inequality follows by taking $g(x)=x$, $a=2$, and $b=1$.
Let A, B be nonnegative real numbers then

$$(A^a - B^a)(A^b - B^b) \geq 0$$

which implies that

$$A^{a+b} + B^{a+b} \geq A^a B^b + A^b B^a.$$

Let $A = f(x)g(y)$ and $B = f(y)g(x)$ then by integrating over $[0,1]\times[0,1]$ we obtain

$$\int_{x=0}^1 \int_{y=0}^1 (f(x)g(y))^{a+b}\,dx\,dy + \int_{x=0}^1 \int_{y=0}^1 (f(y)g(x))^{a+b}\,dx\,dy$$
$$\geq \int_{x=0}^1 \int_{y=0}^1 (f(x)g(y))^a (f(y)g(x))^b\,dx\,dy + \int_{x=0}^1 \int_{y=0}^1 (f(x)g(y))^b (f(y)g(x))^a\,dx\,dy$$

that is

$$\left(\int_0^1 f^{a+b}(x)\,dx\right)\left(\int_0^1 g^{a+b}(y)\,dy\right) + \left(\int_0^1 f^{a+b}(y)\,dy\right)\left(\int_0^1 g^{a+b}(x)\,dx\right)$$
$$\geq \left(\int_0^1 f^a(x)g^b(x)\,dx\right)\left(\int_0^1 f^b(y)g^a(y)\,dy\right) + \left(\int_0^1 f^a(y)g^b(y)\,dy\right)\left(\int_0^1 f^b(x)g^a(x)\,dx\right),$$

and the proof is complete. □