Problem 11815
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Proposed by G. Apostolopoulos (Greece).

Let x, y, and z be positive numbers such that $x+y+z=3$. Prove that

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}\geq 3xyz.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\begin{aligned}\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}&=(x^2-x+1)+(y^2-y+1)+(z^2-z+1)\\&=3\cdot\frac{x^2+y^2+z^2}{3}\geq 3\left(\frac{x+y+z}{3}\right)^2=3\\&=3\left(\frac{x+y+z}{3}\right)^3\geq 3xyz,\end{aligned}$$

where we used first the convexity of $t\to t^2$ and, then the AGM inequality. □