Problem 11814
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Proposed by C. Lupu (USA).

Let ϕ be a continuously differentiable function from $[0,1]$ into $\mathbb{R}$, with $\phi(0)=0$ and $\phi(1)=1$, and suppose that $\phi'(x)\neq 0$ for $0\leq x\leq 1$. Let f be a continuous function from $[0,1]$ into $\mathbb{R}$, such that $\int_0^1 f(x)\,dx=\int_0^1 \phi(x)f(x)\,dx$. Show that there exists t with $0<t<1$ such that $\int_0^t \phi(x)f(x)\,dx=0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$g(s)=\begin{cases}\dfrac{\phi'(s)}{(\phi(s))^2}\displaystyle\int_0^s \phi(x)f(x)\,dx & \text{if } s\in(0,1],\\[2mm] \dfrac{f(0)}{2} & \text{if } s=0\end{cases}$$

(note that by Rolle's theorem $\phi(s)\neq 0$ for $s\in(0,1)$).
The function g is continuous in $[0,1]$ because

$$\lim_{s\to 0^+}\frac{\phi'(s)}{(\phi(s))^2}\int_0^s \phi(x)f(x)\,dx \stackrel{H}{=} \phi'(0)\lim_{s\to 0^+}\frac{\phi(s)f(s)}{2\phi(s)\phi'(s)}=\frac{f(0)}{2}.$$

Let

$$G(t)=\int_0^t g(s)\,ds$$

then $G(0)=0$ and

$$\begin{aligned}G(1)&=\int_0^1 \frac{d}{ds}\left(-\frac{1}{\phi(s)}\right)\left(\int_0^s \phi(x)f(x)\,dx\right)\,ds\\&=\left[\left(-\frac{1}{\phi(s)}\right)\cdot\left(\int_0^s \phi(x)f(x)\,dx\right)\right]_0^1+\int_0^1 \frac{1}{\phi(s)}\frac{d}{ds}\left(\int_0^s \phi(x)f(x)\,dx\right)\,ds\\&=-\frac{1}{\phi(1)}\int_0^1 \phi(x)f(x)\,dx+\lim_{s\to 0^+}\frac{1}{\phi(s)}\int_0^s \phi(x)f(x)\,dx+\int_0^1 f(s)\,ds\\&=-\frac{1}{\phi(1)}\int_0^1 \phi(x)f(x)\,dx+\lim_{s\to 0^+}\frac{\phi(s)f(s)}{\phi'(s)}+\int_0^1 f(s)\,ds\\&=-\frac{1}{\phi(1)}\int_0^1 \phi(x)f(x)\,dx+\frac{\phi(0)f(0)}{\phi'(0)}+\int_0^1 f(s)\,ds=0.\end{aligned}$$

Therefore, by Rolle's theorem, there exists $t\in(0,1)$ such that $G'(t)=g(t)=0$. Since $\phi(t)\neq 0$ and $\phi'(t)\neq 0$ for $t\in(0,1)$, it follows that

$$\int_0^t \phi(x)f(x)\,dx=0.$$

□