Problem 11813
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Proposed by G. Oman (USA).

Let X be a set, and let $*$ be a binary operation on X (that is, a function from $X \times X$ to X). Prove or disprove: there exists an uncountable set X and a binary operation $*$ on X such that for any subsets Y and Z of X that are closed under $*$, either $Y \subseteq Z$ or $Z \subseteq Y$.

Solution proposed by Carlo Pagano, Mathematical Institute, Leiden University, The Netherlands, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

We will show that there exists an uncountable set X with a non-associative binary operation which satisfy the required property. Moroever we will prove that if the binary operation is associative then a set X with that property can not be uncountable.

The non-associative case.
Let X be the *first uncountable ordinal space*, an uncountable well-ordered set such that for any $y \in X$ the set

$$I_y := \{x \in X \ : \ x < y\}$$

is countable. Let x_0 be the smallest element of X. Note that each $x \in X$ has a successor

$$s(x) := \min(X \setminus (I_x \cup \{x\})).$$

If $y \notin s(X)$ and $y \neq x_0$ then I_y is infinite countable and there is an infinite strictly increasing sequence $\{x_k\}_{k\geq 1} \subseteq I_y$ such that $\lim_{k\to\infty} x_k = y$ (i. e. for all $z \in I_y$ there is $n > 0$, such that $z < x_k < y$ for $k > n$). Now we define the function $f : X \times X \to X$ as follows.

i) If $y \in s(X)$ then $y = s(x)$ for some $x \in I_y$. Let $f(y, y) = x$.

ii) If $y \notin s(X)$ and $y \neq x_0$ then $\lim_{k\to\infty} x_k = y$ for some sequence. Let $f(y, y) = x_1$ and let $f(y, x_k) = x_{k+1}$ for $k \geq 1$.

iii) Otherwise, let $f(y, z) = x_0$.

Note that the corresponding binary operation $*$ is not associative because if $y \notin s(X)$ and $y \neq x_0$ then

$$y * (y * x_1) = f(y, f(y, x_1)) = f(y, x_2) = x_3 > f(x_1, x_1) = f(f(y, y), x_1) = (y * y) * x_1.$$

Let Z be a subset closed under $*$. It suffices to show that

$$\bigcup_{x\in Z} I_x \subseteq Z,$$

because if Y is another set closed under $*$ and $z \in Z \setminus Y$ then $Y \subseteq I_z \subseteq Z$ (otherwise there is $y \in Y$ such that $y > z$ and $z \in I_y \subseteq Y$ whereas $z \notin Y$).
Let $Z' := \{x \in Z : I_x \not\subseteq Z\}$ and assume by contradiction that Z' is not empty. Let $z = \min(Z')$ (X is well-ordered) then $z \neq x_0$ and finally we distinguish two cases.

i) If $z \in s(X)$ then $z = s(x)$ for some $x \in I_z$. and $f(z, z) = x \in Z$. Hence, by the minimality of z, $I_x \subseteq Z$ and $I_z = I_x \cup \{x\} \subseteq Z$ which is a contradiction.

ii) If $z \notin s(X)$ then $f(z, z) = x_1 \in Z$, $f(y, x_k) = x_{k+1} \in Z$ for $k \geq 1$. Hence, by the minimality of z, $I_{x_k} \subseteq Z$ for $k \geq 1$ and $\lim_{k\to\infty} x_k = z$ implies that $I_z = \bigcup_{k\geq 1} I_{x_k} \subseteq Z$ which is a contradiction.

$\square$

The associative case.
Assume that X is uncountable. For $x \in X$, let $P(x) = \{x^n : n \geq 1\}$ where $x^n := x * x * \cdots * x$ (n times). Then $P(x)$ is countable and closed under $*$.
Let $a_0 \in X$ then there is $a_1 \in X \setminus P(a_0)$ because X is uncountable. Since the subsets of X which are closed under $*$ are totally ordered by inclusion and $a_1 \notin P(a_0)$, it follows that $P(a_0) \subsetneq P(a_1)$. Continuing in this way, we obtain a sequence $\{a_n : n \geq 0\}$ such that

$$P(a_0) \subsetneq P(a_1) \subsetneq P(a_2) \subsetneq \cdots \qquad \text{with } a_n \notin P(a_{n-1}) \text{ for } n \geq 1.$$

Now the set

$$A = \bigcup_{n \geq 0} P(a_n)$$

is countable and closed under $*$. As before, there is $b \in X \setminus A$ and $A \subsetneq P(b)$.
Hence $a_0 = b^m$ for some $m \geq 1$, which implies that

$$P(b) = \left\{b, b^2, \ldots, b^{m-1}\right\} \cup P(a_0) \cup b * P(a_0) \cup b^2 * P(a_0) \cup \cdots \cup b^{m-1} * P(a_0).$$

Since $\left\{b, b^2, \ldots, b^{m-1}\right\}$ is finite and $a_n \notin P(a_0)$ for $n > 0$ there is $0 < k < m$ such that $b^k * P(a_0)$ contains infinite elements of $\{a_n : n \geq 1\}$. Let a_{n_1} be one of them

$$a_{n_1} = b^k * a_0^{j_1} \quad \text{for some } j_1 \geq 1.$$

Among the other infinite elements, there is a_{n_2} with $n_2 > n_1$ such that

$$a_{n_2} = b^k * a_0^{j_2} \quad \text{for some } j_2 > j_1.$$

Since $a_0 \in P(a_{n_1})$, it follows that $a_0 = a_{n_1}^r$ for some $r \geq 1$ and therefore

$$a_{n_2} = b^k * a_0^{j_2} = a_{n_1} * a_0^{j_2 - j_1} = a_{n_1}^{1 + r(j_2 - j_1)}$$

which contradicts $a_{n_2} \notin P(a_{n_1})$. $\square$

Note that there exists $(X, *)$ where X is countably infinite: take

$$X = \{\exp(2\pi i k / p^n) \mid k \in \mathbf{Z},\, n \in \mathbf{Z}^+\}.$$

where p is a prime and the binary operation $*$ is the multiplication of complex numbers. It is easy to see that its subgroups are totally ordered by inclusion.