Problem 11812
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Proposed by C. Chiser (Romania).

Let f be a twice continuously differentiable function from $[0,1]$ into $\mathbb{R}$. Let n be an integer greater than 1. Given that $\sum_{k=1}^{n-1} f(k/n) = -(f(0)+f(1))/2$, prove that

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{5!n^4}\int_0^1 (f''(x))^2\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By the second-order Euler-Maclaurin Formula, we have that for any $g \in C^2([0,n])$,

$$\sum_{k=0}^{n-1} g(k) = \int_0^n g(x)\,dx + B_1(g(n)-g(0)) + \frac{B_2}{2}(g'(n)-g'(0)) - \frac{1}{2}\int_0^n B_2(\{x\})g''(x)\,dx$$

where B_i is the ith Bernoulli number ($B_1 = 1/2$, $B_2 = 1/6$), $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial, and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.
Hence, by taking $g(x) = f(x/n)$, we get

$$\sum_{k=0}^{n-1} f(k/n) = n\int_0^1 f(x)\,dx + \frac{f(1)-f(0)}{2} + \frac{f'(1)-f'(0)}{12n} - \frac{1}{2n}\int_0^1 B_2(\{nx\})f''(x)\,dx$$

and by using $\sum_{k=1}^{n-1} f(k/n) = -(f(0)+f(1))/2$, we obtain

$$\begin{aligned}\int_0^1 f(x)\,dx &= \frac{1}{2n^2}\int_0^1 B_2(\{nx\})f''(x)\,dx - \frac{f'(1)-f'(0)}{12n^2}\\ &= \frac{1}{2n^2}\int_0^1 \left(B_2(\{nx\}) - \frac{1}{6}\right) f''(x)\,dx\\ &= \frac{1}{2n^2}\int_0^1 \left(\{nx\}^2 - \{nx\}\right) f''(x)\,dx.\end{aligned}$$

Finally, by Cauchy-Schwarz inequality,

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{4n^4}\int_0^1 \left(\{nx\}^2 - \{nx\}\right)^2\,dx \cdot \int_0^1 (f''(x))^2\,d = \frac{1}{120n^4}\int_0^1 (f''(x))^2\,dx,$$

because

$$\int_0^1 \left(\{nx\}^2 - \{nx\}\right)^2\,dx = \frac{1}{n}\int_0^n \left(\{x\}^2 - \{x\}\right)^2\,dx = \int_0^1 \left(x^2 - x\right)^2\,dx = \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1 = \frac{1}{30}.$$

□