Problem 11811
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Proposed by Vazgen Mikayelyan (Armenia).

Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be infinite sequences of positive numbers. Let $\{x_n\}_{n\geq 1}$ be the infinite sequence given for $n \geq 1$ by

$$x_n = \frac{a_1^{b_1} \cdots a_n^{b_n}}{\left(\frac{a_1 b_1+\cdots+a_n b_n}{b_1+\cdots+b_n}\right)^{b_1+\cdots+b_n}}.$$

(a) Prove that $\lim_{n\to\infty} x_n$ exists.
(b) Find the set of all c that can occur as that limit, for suitably chosen $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

(a) Let $B_n = b_1 + \cdots + b_n$. It suffices to show that $\{x_n\}_{n\geq 1}$ is decreasing sequence of positive numbers. We have that $x_1 = 1$ and for $n \geq 1$

$$\frac{x_{n+1}}{x_n} = \frac{a_{n+1}^{b_{n+1}} \left(\frac{a_1 b_1+\cdots+a_n b_n}{B_n}\right)^{B_n}}{\left(\frac{a_1 b_1+\cdots+a_{n+1} b_{n+1}}{B_{n+1}}\right)^{B_{n+1}}} \leq 1$$

if and only if

$$a_{n+1}^{b_{n+1}} \left(\frac{a_1 b_1+\cdots+a_n b_n}{B_n}\right)^{B_n} \leq \left(\frac{a_1 b_1+\cdots+a_{n+1} b_{n+1}}{B_{n+1}}\right)^{B_{n+1}},$$

or

$$\alpha \ln(x) + (1-\alpha)\ln(y) \leq \ln(\alpha x + (1-\alpha) y)$$

where $\alpha = b_{n+1}/B_{n+1}$, $x = a_{n+1}$ and $y = (a_1 b_1 + \cdots + a_n b_n)/B_n$. The last inequality holds because the natural logarithm is concave in $(0, +\infty)$.

(b) Note that by (a) all the limit points belong to $[0, 1]$. We show that $[0, 1]$ is precisely the set of limit points.
If $a_n = n$ and $b_n = 1$ then

$$x_n = \frac{n!}{\left(\frac{1+\cdots+n}{n}\right)^n} = \frac{n!}{\left(\frac{n+1}{2}\right)^n} \to 0.$$

Finally, if if $a_n = 1 + t/n$ and $b_n = 1$ with $t \geq 0$, then

$$x_n = \frac{\prod_{k=1}^n \left(1+\frac{t}{k}\right)}{\left(1+\frac{tH_n}{n}\right)^n} \to \frac{1}{t\Gamma(t)e^{\gamma t}} = \frac{1}{\Gamma(t+1)e^{\gamma t}}$$

where $H_n = \sum_{k=1}^n 1/k = \ln n + \gamma + O(1/n)$.
By varying $t \in [0, +\infty)$, the continuous function $1/(\Gamma(t+1)e^{\gamma t})$ covers up the interval $(0, 1]$. □