Problem 11810
(American Mathematical Monthly, Vol.122, January 2015)

Proposed by O. Furdui (Romania).

Find

$$\sum_{n=1}^{\infty} H_n \left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3}\right)$$

where $H_n = \sum_{k=1}^{n} 1/k$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\begin{aligned}\sum_{n=1}^{\infty} H_n \left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3}\right) = \sum_{n=1}^{\infty} H_n \sum_{k=n+1}^{\infty} \frac{1}{k^3} &= \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{n=1}^{k-1} H_n = \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{n=1}^{k-1} \sum_{j=1}^{n} \frac{1}{j} \\ &= \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{j=1}^{k-1} \frac{1}{j} \sum_{n=j}^{k-1} 1 = \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{j=1}^{k-1} \frac{k-j}{j} \\ &= \sum_{k=2}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k-1} \frac{1}{j} - \sum_{k=2}^{\infty} \frac{k-1}{k^3} \\ &= \zeta(2,1) - \zeta(2) + \zeta(3) = 2\zeta(3) - \zeta(2)\end{aligned}$$

where in the last step we used the famous identity of Euler $\zeta(2,1) = \zeta(3)$. □