Problem 11809
(American Mathematical Monthly, Vol.121, December 2014)

Proposed by O. Kouba (Syria).

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

(a) Suppose $\{a_n\}_{n\geq 1}$ consists of nonnegative numbers and is nonincreasing, and $\sum_{n=1}^{\infty} a_n/\sqrt{n}$ converges. Prove that $\sum_{n=1}^{\infty}(-1)^{\lfloor\sqrt{n}\rfloor}a_n$ converges.

(b) Find a nonincreasing sequence $\{a_n\}_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty}\sqrt{n}a_n = 0$ and $\sum_{n=1}^{\infty}(-1)^{\lfloor\sqrt{n}\rfloor}a_n$ diverges.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, Italy.

(a) We first note that $\lim_{n\to\infty}\sqrt{n}a_n = 0$. Otherwise there exist $\varepsilon > 0$ and $n_0 \geq 1$, such that for all $n \geq n_0$, $\sqrt{n}a_n > \varepsilon$ and

$$\sum_{k=n_0}^{n}\frac{a_k}{\sqrt{k}} > \varepsilon\sum_{k=n_0}^{n}\frac{1}{k},$$

which contradicts the fact that $\sum_{n=1}^{\infty} a_n/\sqrt{n}$ converges.
Moreover, we have that

$$S_n := \sum_{k=1}^{n}(-1)^{\lfloor\sqrt{k}\rfloor} = \sum_{k=1}^{a-1}(-1)^k((k+1)^2-k^2)+(-1)^a(n+1-a^2) = -(-1)^a a-1+(-1)^a(n+1-a^2)$$

where $a = \lfloor\sqrt{n}\rfloor$. Therefore $|S_n| \leq |n-a^2-a|+2 \leq a+2 \leq \sqrt{n}+2$.
Now, for $1 \leq n \leq m$,

$$\sum_{k=n}^{m}(-1)^{\lfloor\sqrt{k}\rfloor}a_k = \sum_{k=n}^{m}(S_k-S_{k-1})a_k = \sum_{k=n}^{m-1}S_k(a_k-a_{k+1}) - S_{n-1}a_n + S_m a_m.$$

Since $\{a_n\}_{n\geq 1}$ is a nonincreasing sequence, we have

$$\begin{aligned}\sum_{k=n}^{m-1}|S_k(a_k-a_{k+1})| &\leq \sum_{k=n}^{m-1}(\sqrt{k}+2)(a_k-a_{k+1})\\ &= \sum_{k=n}^{m}(\sqrt{k}-\sqrt{k-1})a_k + (\sqrt{n-1}+2)a_n - (\sqrt{m}+2)a_m\\ &\leq \sum_{k=n}^{m}\frac{a_k}{\sqrt{k}} + (\sqrt{n-1}+2)a_n - (\sqrt{m}+2)a_m.\end{aligned}$$

Hence, it follows

$$\left|\sum_{k=n}^{m}(-1)^{\lfloor\sqrt{k}\rfloor}a_k\right| \leq \sum_{k=n}^{m-1}|S_k(a_k-a_{k+1})| + |S_{n-1}|a_n + |S_m|a_m \leq \sum_{k=n}^{m}\frac{a_k}{\sqrt{k}} + 2(\sqrt{n-1}+2)a_n$$

which implies that $\{\sum_{k=1}^{n}(-1)^{\lfloor\sqrt{k}\rfloor}a_k\}_{n\geq 1}$ is a Cauchy sequence because each term on the right hand side goes to zero as n and m go to infinity.

(b) For $n \geq 1$, let

$$a_n = \frac{1}{x_n \ln(1 + x_n)} \qquad \text{with} \quad x_n = \lfloor (\sqrt{n} + 1)/2 \rfloor.$$

Then $\{a_n\}_{n\geq 1}$ is a nonincreasing sequence of positive numbers such that for $n > 1$,

$$0 < \sqrt{n} a_n < \frac{2\sqrt{n}}{(\sqrt{n} - 1) \ln((\sqrt{n} + 1)/2)}$$

because $x - 1 < \lfloor x \rfloor$. Hence $\lim_{n\to\infty} \sqrt{n} a_n = 0$. Moreover

$$\begin{aligned} (-1)^{\lfloor \sqrt{n} \rfloor} &= -1 \quad \text{and} \quad x_n = m \quad \text{for } (2m-1)^2 \leq n < (2m)^2, \\ (-1)^{\lfloor \sqrt{n} \rfloor} &= +1 \quad \text{and} \quad x_n = m \quad \text{for } (2m)^2 \leq n < (2m+1)^2, \end{aligned}$$

which implies that for $N < (2m + 1)^2$,

$$\sum_{n=1}^{N} (-1)^{\lfloor \sqrt{n} \rfloor} a_n \geq \sum_{n=1}^{m-1} \frac{2}{n \ln(1 + n)} - \frac{(4m - 1)}{m \ln(1 + m)}.$$

Therefore the series diverges

$$\sum_{n=1}^{\infty} (-1)^{\lfloor \sqrt{n} \rfloor} a_n \geq \sum_{n=1}^{\infty} \frac{2}{n \ln(1 + n)} - \lim_{m\to\infty} \frac{(4m - 1)}{m \ln(1 + m)} = +\infty.$$

□