Problem 11808
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Proposed by D.M. Batinetu-Giurgiu (Romania) and N. Stanciu (Romania).

Compute

$$\lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} \Gamma(nx)\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that if f is a continuous real function in (a,b) and $e\in(a,b)$ then

$$\lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx)\,dx = ef(e).$$

In our case Γ is a continuous real function in $(0,+\infty)$ and therefore the required limit is $e\Gamma(e)$.
Let $b_n = n(n!)^{-1/n}$ and $a_n = n((n+1)!)^{-1/(n+1)}$, then by the Mean Value Theorem for integrals,

$$n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx)\,dx = n\int_{a_n}^{b_n} f(t)\,dt = n(b_n-a_n)f(t_n)$$

for some $t_n\in(a_n,b_n)$. Now, by the Stirling approximation formula,

$$\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(n) + \ln(\sqrt{2\pi}) + O(1/n).$$

Hence

$$\begin{aligned} b_n &= n\exp(-\ln(n!)/n) = e - \frac{e\ln(n)}{2n} - \frac{e\ln(\sqrt{2\pi})}{n} + O(\ln^2(n)/n^2),\\ b_n - a_n &= b_n - \frac{nb_{n+1}}{n+1} = \frac{e}{n} + O(\ln(n)/n^2), \end{aligned}$$

which imply that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} t_n = e$$

and, by the continuity of f at e,

$$\lim_{n\to\infty} n(b_n-a_n)f(t_n) = ef(e).$$

□