Problem 11801
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Proposed by D. Carter (USA).

Let f be a polynomial in one variable with rational coefficients that has no nonnegative real root. Show that there is a nonzero polynomial g with rational coefficients such that the coefficients of $f \cdot g$ are positive.

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By continuity, we may assume that f is positive in $[0, +\infty)$ (otherwise we replace f and g with $(-f)$ and $(-g)$). We will show that for any polynomial $f \in \mathbb{R}[x]$ which is positive in $[0, +\infty)$, there exists a nonnegative integer M such that the coefficients of $(x+1)^M f(x)$ are positive. Note that if the coeffiecients of f are rationals (or integers) the same holds for the product $(x+1)^M f(x)$ (see the similar problem 4441, Am. Math. Monthly 59, 663–664 (1952)).

We factorize f into quadratic and linear factors in $\mathbb{R}[x]$: there exist $a > 0$, $a_i > 0$ for $i = 1, \ldots, r$, and $b_j \in \mathbb{R}$, $c_j > 0$ with $b_j^2 < 4c_j$ for $j = 1, \ldots, s$ such that

$$f(x) = a \prod_{i=1}^{r} (x + a_i) \prod_{j=1}^{s} (x^2 + b_j x + c_j).$$

It is easy to see that if the coefficients of two polynomials are positive then the same holds for their product. So it suffices to solve the problem for the irreducible quadratic factors $x^2 + bx + c$.
For $k = 0, \ldots, m+2$, the k-th coefficient of $(x+1)^m(x^2 + bx + c)$ is

$$\binom{m}{k-2} + b\binom{m}{k-1} + c\binom{m}{k}.$$

The 0-th coefficient is $c > 0$ and $(m+2)$-th coefficient is $1 > 0$. Moreover, for $1 \leq k \leq m+1$ the coefficient is positive iff

$$\begin{aligned} 0 &< \frac{\binom{m}{k-2}}{\binom{m}{k-1}} + b + \frac{c\binom{m}{k}}{\binom{m}{k-1}} = \frac{k-1}{m+2-k} + b + \frac{c(m+1-k)}{k} \\ &= \frac{(1-b+c)k^2 + (bm - 1 + 2b - 2cm - 3c)k + c(m^2 + 3m + 2)}{k(m+2-k)} = \frac{N(k)}{k(m+2-k)} \end{aligned}$$

where

$$N(x) = (1-b+c)x^2 + (bm - 1 + 2b - 2cm - 3c)x + c(m^2 + 3m + 2).$$

Since $x^2 + bx + c$ is irriducible, $b^2 < 4c$ and $x^2 + bx + c$ is always positive. So for $x = -1$ we have $(1 - b + c) > 0$ which implies that the quadratic function $N(x)$ attains the minimum value at $x_0 = -(bm - 1 + 2b - 2cm - 3c)/(2 - 2b + 2c)$. By direct computation we get

$$N(x_0) = \frac{(4c - b^2)m^2 + (2bc + 2b + 8c - 4b^2)m - 4b^2 + 4bc - c^2 + 4b + 2c - 1}{4(1 - b + c)}$$

and for a sufficiently large value of m, $N(x_0) > 0$ because $(4c - b^2)/(1 - b + c) > 0$.
Finally, since we are able to find m_j such that the coefficients of $(x+1)^{m_j}(x^2 + b_j x + c_j)$ are positive then the coefficients of $(x+1)^M f(x)$ are positive by letting $M = \sum_{j=1}^{s} m_j$.