Problem 11798
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Proposed by F. Holland (Ireland).

For positive integers n, let f_n be the polynomial given by

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{\lfloor k/2 \rfloor}.$$

(a) Prove that if $n+1$ is prime, then f_n is irriducible over $\mathbb{Q}$.
(b) Prove for all n,

$$f_n(1+x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k} x^k.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

(a) We have that

$$f_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\binom{n}{2k} + \binom{n}{2k+1} \right) x^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} x^k.$$

If $p = 2$ then $f_{p-1}(x) = 2$ which is irriducible. If $p > 2$ is a prime then

$$f_{p-1}(x) = \sum_{k=0}^{(p-1)/2} \binom{p}{2k+1} x^k = x^{(p-1)/2} + \sum_{k=1}^{(p-1)/2} \binom{p}{2k+1} x^k + p.$$

Since the leading coefficient is 1, p divides the coefficients $\binom{p}{2k+1}$ for $k = 1, \ldots, (p-1)/2$ and $f_{p-1}(0) = p$, it follows by Eisenstein's criterion that f_{p-1} is irriducible over $\mathbb{Q}$.

(b) The polynomial

$$P_n(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k} x^k$$

satisfies the linear recurrence

$$P_0(x) = 1,\ P_1(x) = 2,\ P_n(x) = 2P_{n-1}(x) + xP_{n-2}(x) \text{ for } n \geq 2.$$

Hence, for $x > -1$

$$P_n(x) = \frac{u^{n+1} - v^{n+1}}{u - v}$$

where $u = 1 + \sqrt{1+x}$, $v = 1 - \sqrt{1+x}$ are the solutions of the equation $z^2 = 2z + x$. Moreover,

$$\begin{aligned} \frac{u^{n+1} - v^{n+1}}{u - v} &= \frac{1}{2\sqrt{1+x}} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} (1+x)^{k/2} - \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (1+x)^{k/2} \right) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} (1+x)^k = f_n(1+x) \end{aligned}$$

Hence the required identity holds for $x > -1$. Since we are comparing two polynomials, it follows that the identity holds for all $x \in \mathbb{C}$.