Problem 11796
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Proposed by G. Glebov (Canada) and S. Fraser (Canada).

Find

$$\int_0^\infty \frac{\sin((2n+1)x)}{\sin(x)}\, e^{-\alpha x} x^{m-1}\, dx$$

in terms of α, m, and n, when $\alpha > 0$, $m \geq 1$, and n is a nonnegative integer.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\begin{aligned} f_n(x) = \frac{\sin((2n+1)x)}{\sin(x)} &= \frac{\sin((2n-1)x)\cos(2x) + \cos((2n-1)x)\sin(2x)}{\sin(x)} \\ &= f_{n-1}(x)(1 - 2\sin^2(x)) + 2\cos((2n-1)x)\cos(x) \\ &= f_{n-1}(x) + 2(\cos((2n-1)x)\cos(x) - \sin((2n-1)x)\sin(x)) \\ &= f_{n-1}(x) + 2\cos(2nx) = 1 + 2\sum_{k=1}^{n}\cos(2kx) = 1 + 2\sum_{k=1}^{n}\mathrm{Re}(e^{i2kx}). \end{aligned}$$

Since the Laplace transform of x^{m-1} is $\mathcal{L}[x^{m-1}](z) = \Gamma(m)z^{-m}$, it follows

$$\begin{aligned} \int_0^\infty \frac{\sin((2n+1)x)}{\sin(x)}\, e^{-\alpha x} x^{m-1}\, dx &= \int_0^\infty \left(1 + 2\sum_{k=1}^{n}\mathrm{Re}(e^{i2kx})\right) e^{-\alpha x} x^{m-1}\, dx \\ &= \mathcal{L}[x^{m-1}](\alpha) + 2\mathrm{Re}\left(\sum_{k=1}^{n}\mathcal{L}[x^{m-1}](\alpha - 2ik)\right) \\ &= \Gamma(m)\left(\alpha^{-m} + 2\mathrm{Re}\left(\sum_{k=1}^{n}(\alpha - 2ik)^{-m}\right)\right). \end{aligned}$$

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