Problem 11795
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Proposed by M. Merca (Romania).

Let p be the partition counting function and let g be the function given by $g(n) = \frac{1}{2}\lceil n/2\rceil\lceil(3n+1)/2\rceil$. Let $A(n)$ be the set of non-negative integer triples (i, j, k) such that $g(i) + j + k = n$. Prove that for $n \geq 1$,

$$p(n) = \frac{1}{n} \sum_{(i,j,k)\in A(n)} (-1)^{\lceil i/2\rceil -1} g(i)p(j)p(k).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By the pentagonal number theorem

$$G(x) := \prod_{n=1}^{\infty}(1 - x^n) = \sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil} x^{g(n)}.$$

Let P be the generating function for $p(n)$, then

$$P(x) = \sum_{n=1}^{\infty} p(n)x^n = \frac{1}{G(x)},$$

Moreover,

$$xG'(x) = \sum_{n=1}^{\infty}(-1)^{\lceil n/2\rceil} g(n)x^{g(n)} = -G(x)\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}$$

and

$$xP'(x) = \sum_{n=1}^{\infty} np(n)x^n = \frac{1}{G(x)}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}.$$

Hence $xP'(x) = -xG'(x)(P(x))^2$ and by extracting the coefficient of x^n for $n \geq 1$, we obtain

$$np(n) = -\sum_{(i,j,k)\in A(n)} (-1)^{\lceil i/2\rceil} g(i)p(j)p(k).$$

where

$$A(n) := \{(i, j, k) \in \mathbb{N}^3 \ : \ g(i) + j + k = n\}.$$

□

Note that in the original statement

$$A(n) := \{(i, j, k) \in (\mathbb{N}^+)^3 \ : \ g(i) + j + k = n\}$$

but in this case the formula does not work because j and k do not assume the value 0 (recall that $p(0) = 1$).