Problem 11793
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Prove that

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^2} = -\zeta'(2) + \sum_{n=3}^{\infty} (-1)^{n+1} \frac{\zeta(n)}{n-2}.$$

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It is well known that for $x \in (-1, 1]$,

$$\Psi(x+1) + \gamma = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{x+n} \right) = \sum_{n=2}^{\infty} (-1)^n x^{n-1} \zeta(n),$$

where $\Psi(x) := \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Gamma function.
Hence,

$$-\sum_{k=1}^{\infty} \frac{1}{n^2(x+n)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{x+n} \right) \frac{1}{x^2} - \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{k^2} = \sum_{n=3}^{\infty} (-1)^n x^{n-3} \zeta(n).$$

By integrating over $[0, 1]$, we get

$$-\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^2} + \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} = \sum_{n=3}^{\infty} (-1)^n \frac{\zeta(n)}{n-2}$$

which is equivalent to the required identity because

$$\zeta'(x) = \frac{d}{dx} \left(\sum_{n=1}^{\infty} \frac{1}{n^x} \right) = -\sum_{n=1}^{\infty} \frac{\ln(n)}{n^x}$$

and therefore

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} = -\zeta'(2).$$

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