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Show that every infinite dimensional Banach space contains a closed subspace of infinite dimension and infinite codimension.

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It is known (see for example Theorem 1.a.5 in *Banach Spaces* by Lindenstrauss and Tzafriri) that every infinite dimensional Banach space X contains a *basic sequence* $\{x_n\}_{n\geq 1}$ which means that for all x in the closed linear span of $\{x_n\}_{n\geq 1}$ there is a unique sequence of scalars $\{a_n\}_{n\geq 1}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

Now it is immediate that the closed linear span of $\{x_{2n}\}_{n\geq 1}$ is a closed subspace of infinite dimension and infinite codimension as required.

For the sake of completeness we give the details of the basic sequence's construction due to S. Mazur.

i) Let F be a finite-dimensional subspace of X, and let $\epsilon > 0$, then there exists $x \in X$ such that $\|x\| = 1$ and $\|y\| \leq (1+\epsilon)\|y + ax\|$ for all $y \in F$, and for all scalars a.

   We may assume that $\epsilon < 1$ and $\|y\| = 1$. As the unit ball of F is compact, there is a finite set of unit vectors $y_1, y_2, \ldots, y_d$ such that for all unit vectors $y \in F$, there is a y_k with $k \in [1, d]$, such that $\|y - y_k\| < \epsilon/2$. Let $y'_1, y'_2, \ldots, y'_d$ be unit vectors in the dual of X such that $\langle y'_k, y_k\rangle = 1$ for $k \in [1, d]$. Then there is $x \in X$ with $\|x\| = 1$ and $\langle y'_k, x\rangle = 0$ for $k \in [1, d]$. Hence for every scalar a,

$$\|y + ax\| \geq \|y_k + ax\| - \|y - y_k\| \geq |\langle y'_k, y_k + ax\rangle| - \epsilon/2 = 1 - \epsilon/2 \geq \frac{1}{1+\epsilon}.$$

ii) Let x_1 be a unit vector in X. For $n > 1$, by i), there is a unit vector $x_n \in X$ such that $\|y\| \leq (1 + 1/2^n)\|y + ax_n\|$ for all y in the linear span of $\{x_1, x_2, \ldots, x_{n-1}\}$, and for all scalars a.

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