Problem 11791
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Show that for $n \geq 1$,

$$\sum_{k=1}^{n} \binom{6n+1}{6k-2} B_{6k-2} = -\frac{6n+1}{6},$$

where B_n denotes the nth Bernoulli number.

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The generating function of the Bernoulli numbers is

$$f(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}.$$

Let $\omega = e^{2\pi i/3}$, and let $F(z) := f(z) + \omega^2 f(\omega z) + \omega f(\omega^2 z)$. Then

$$F(z) = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!} \left(1 + \omega^{n+2} + \omega^{2n+1}\right) = 3 \sum_{n=0}^{\infty} \frac{B_{3n+1} z^{3n+1}}{(3n+1)!} = -\frac{3z}{2} + 3 \sum_{n=1}^{\infty} \frac{B_{6n-2} z^{6n-2}}{(6n-2)!},$$

where in the last step we used the fact that $B_1 = -1/2$ and $B_n = 0$ for any odd integer $n > 1$. On the other hand, letting

$$g(z) := e^z + e^{\omega z} + e^{\omega^2 z} = 3 \sum_{n=0}^{\infty} \frac{z^{3n}}{(3n)!} = 3 \sum_{n=0}^{\infty} \frac{z^{6n}}{(6n)!} + 3 \sum_{n=0}^{\infty} \frac{z^{6n+3}}{(6n+3)!},$$

we get

$$\begin{aligned} F(z) &= z \left(\frac{1}{e^z - 1} + \frac{1}{e^{\omega^2 z} - 1} + \frac{1}{e^{\omega z} - 1} \right) \\ &= \frac{z(3 + e^{-z} - 2e^z + e^{-\omega z} - 2e^{\omega z} + e^{-\omega^2 z} - 2e^{\omega^2 z})}{e^z - e^{-z} + e^{\omega z} - e^{-\omega z} + e^{\omega^2 z} - e^{-\omega^2 z}} \\ &= \frac{z(3 - 2g(z) + g(-z))}{g(z) - g(-z)} = \frac{-3 \displaystyle\sum_{n=1}^{\infty} \frac{z^{6n+1}}{(6n)!} - 9 \sum_{n=0}^{\infty} \frac{z^{6n+4}}{(6n+3)!}}{6 \displaystyle\sum_{n=0}^{\infty} \frac{z^{6n+3}}{(6n+3)!}}. \end{aligned}$$

Therefore, we obtain the following identity

$$6 \sum_{n=0}^{\infty} \frac{z^{6n+3}}{(6n+3)!} \cdot \left(-\frac{3z}{2} + 3 \sum_{n=1}^{\infty} \frac{B_{6n-2} z^{6n-2}}{(6n-2)!} \right) = -3 \sum_{n=1}^{\infty} \frac{z^{6n+1}}{(6n)!} - 9 \sum_{n=0}^{\infty} \frac{z^{6n+4}}{(6n+3)!},$$

that is

$$\sum_{n=0}^{\infty} \frac{z^{6n+3}}{(6n+3)!} \cdot \sum_{n=1}^{\infty} \frac{B_{6n-2} z^{6n-2}}{(6n-2)!} = -\frac{1}{6} \sum_{n=1}^{\infty} \frac{(6n+1) z^{6n+1}}{(6n+1)!},$$

and after extracting the coefficient of $z^{6n+1}/(6n+1)!$, we finally have

$$\sum_{k=1}^{n} \binom{6n+1}{6k-2} B_{6k-2} = -\frac{6n+1}{6}.$$

□