Problem 11789
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Let a and b be positive integers. Prove that for every positive integer m there exists a positive integer n such that m divides $ba^n + n$.

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We will show a more general statement.
Let a and b be positive integers. If m is a positive integer then for any $j = 0, 1, \ldots, m-1$, there exists an arbitrarily large positive integer $n_j(m)$ such that

$$ba^{n_j(m)} + n_j(m) \equiv j \pmod{m}.$$

We proceed by induction on m. If $m = 1$ then it trivially holds. For $m = 2$, let $n_j(2) \equiv j - ba \pmod{2}$. Now assume that the statement holds for $1, 2\ldots, m-1$. Due to Euler-Fermat theorem, the sequence $\{a^n\}_{n\geq 1}$ is eventually periodic modulo m. Let T be the least positive integer such that $a^n \equiv a^{n+T} \pmod{m}$ for all sufficiently large integer n. Note that $T \leq \varphi(m) < m$, because $a^{n+\varphi(m)} \equiv a^n \pmod{m}$ for all $n \geq m - \varphi(m)$.
Let $d = \gcd(m, T) \leq \min(m, T) < m$. Then, by the inductive hypothesis, we are able to consider the following set of positive integers

$$\{ba^{n_j(d)+kT} + (n_j(d) + kT) \ : \ j = 0, \ldots, d-1, \ k = 0, \ldots, m/d - 1\}.$$

It suffices to show that these integers are all distinct modulo m and therefore they cover all the $d(m/d) = m$ possible remainders.
If for some $0 \leq k_1 \leq k_2 < m/d$, and for some $0 \leq j_1 \leq j_2 < d$, we have

$$ba^{n_{j_1}(d)+k_1T} + (n_{j_1}(d) + k_1T) \equiv ba^{n_{j_2}(d)+k_2T} + (n_{j_2}(d) + k_2T) \pmod{m},$$

then

$$(ba^{n_{j_1}(d)} + n_{j_1}(d)) - (ba^{n_{j_2}(d)} + n_{j_2}(d)) \equiv (k_2 - k_1)T \pmod{m}.$$

i) If $j_1 < j_2$, then $0 < j_2 - j_1 < d$ and we obtain the following contradiction

$$0 \not\equiv j_1 - j_2 \equiv (ba^{n_{j_1}(d)} + n_{j_1}(d)) - (ba^{n_{j_2}(d)} + n_{j_2}(d)) \equiv (k_2 - k_1)T \equiv 0 \pmod{d}.$$

ii) On the other hand, if $j_1 = j_2$ and $k_1 < k_2$, then

$$(k_2 - k_1)T \equiv 0 \pmod{m}.$$

which implies that the positive integer $(k_2 - k_1)T$ is a multiple of both T and m, which is impossible because $(k_2 - k_1)T < (m/d)T = \mathrm{lcm}(m, T)$.

□