Problem 11788
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Proposed by S. P. Andriopoulos (Greece).

Let n be a positive integer, and suppose that $0 < y_i \leq x_i < 1$ for $1 \leq i \leq n$. Prove that

$$\frac{\ln x_1 + \cdots + \ln x_n}{\ln y_1 + \cdots + \ln y_n} \leq \sqrt{\frac{1-x_1}{1-y_1} + \cdots + \frac{1-x_n}{1-y_n}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We show by induction on n that a more general inequality holds: if $0 < y_i \leq x_i < 1$ for $1 \leq i \leq n$ and $t \in [0,1]$ then

$$\frac{\ln x_1 + \cdots + \ln x_n}{\ln y_1 + \cdots + \ln y_n} \leq \left(\frac{1-x_1}{1-y_1} + \cdots + \frac{1-x_n}{1-y_n}\right)^t.$$

For $n = 1$, since $0 < y_1 < 1$ then $\ln y_1 < 0$ and the inequality is equivalent to

$$\frac{\ln x_1}{(1-x_1)^t} \geq \frac{\ln y_1}{(1-y_1)^t}$$

which holds because the map $x \to (\ln x)/(1-x)^t$ is increasing in $(0,1)$:

$$D_x\left(\frac{\ln x}{(1-x)^t}\right) = \frac{1}{x(1-x)^t} + \frac{t\ln x}{(1-x)^{t+1}} = \frac{(1/x) - 1 - t\ln(1/x)}{(1-x)^{t+1}} > 0$$

where we used the fact that $1 < \ln s < s - 1$ for $s > 1$.
Now we prove the induction step. Since $0 < y_n y_{n+1} \leq x_n x_{n+1} < 1$, by the inductive hypothesis,

$$\frac{\ln x_1 + \cdots + \ln x_n + \ln x_{n+1}}{\ln y_1 + \cdots + \ln y_n + \ln y_{n+1}} = \frac{\ln x_1 + \cdots + \ln(x_n x_{n+1})}{\ln y_1 + \cdots + \ln(y_n y_{n+1})} \leq \left(\frac{1-x_1}{1-y_1} + \cdots + \frac{1-x_n x_{n+1}}{1-y_n y_{n+1}}\right)^t.$$

Since $x \to x^t$ is increasing in $[0,+\infty)$, it suffices to show that

$$\frac{1-x_n x_{n+1}}{1-y_n y_{n+1}} \leq \frac{1-x_n}{1-y_n} + \frac{1-x_{n+1}}{1-y_{n+1}}$$

which holds because the linear map

$$x \to \frac{1-x_n}{1-y_n} + \frac{1-x}{1-y_{n+1}} - \frac{1-x_n x}{1-y_n y_{n+1}} = \frac{1-x_n}{1-y_n} + \frac{y_{n+1}(1-y_n)}{(1-y_{n+1})(1-y_n y_{n+1})} - x \cdot \frac{(1-x_n) + y_{n+1}(x_n - y_n)}{(1-y_{n+1})(1-y_n y_{n+1})}$$

is decreasing in $[y_{n+1}, 1]$ and positive at 1. □