Problem 11782
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Proposed by M. Merca (Romania).

Prove that

$$\sum_{k=1}^{\infty}(-1)^{k-1}kp_k(n-k(k+1)/2)=\sum_{k=-\infty}^{\infty}(-1)^k\tau(n-k(3k-1)/2).$$

Here, $p_k(n)$ denotes the number of partitions of n in which the greatest part is less than or equal to k, and n is the number of divisors of n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By the q-binomial theorem

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}z^k q^{\binom{k+1}{2}}=\prod_{k=1}^{n}(1+zq^k)\qquad\text{where}\quad\begin{bmatrix}n\\k\end{bmatrix}=\frac{\prod_{j=n-k+1}^{n}(1-q^j)}{\prod_{j=1}^{k}(1-q^j)}.$$

Let $|z|<1$, then by taking the limit as n goes to infinity, we obtain

$$\sum_{k=0}^{\infty}\frac{z^k q^{\binom{k+1}{2}}}{\prod_{j=1}^{k}(1-q^j)}=\prod_{k=1}^{\infty}(1+zq^k).$$

Now we take the derivative of both sides with respect to z,

$$\sum_{k=1}^{\infty}\frac{kz^{k-1}q^{\binom{k+1}{2}}}{\prod_{j=1}^{k}(1-q^j)}=\prod_{k=1}^{\infty}(1+zq^k)\sum_{k=1}^{\infty}\frac{q^k}{1+zq^k},$$

and, as z goes to -1, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}kq^{\binom{k+1}{2}}}{\prod_{j=1}^{k}(1-q^j)}=\prod_{k=1}^{\infty}(1-q^k)\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}.$$

By using the pentagonal number theorem, we can write the above identity in the following way

$$\sum_{k=1}^{\infty}(-1)^{k-1}kq^{\binom{k+1}{2}}\sum_{j=0}^{\infty}p_k(j)q^j=\left(\sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k-1)}{2}}\right)\cdot\left(\sum_{k=1}^{\infty}\tau(k)q^k\right).$$

Finally, by extracting the coefficient of x^n of both sides, we get

$$\sum_{k=1}^{\infty}(-1)^{k-1}kp_k(n-k(k+1)/2)=\sum_{k=-\infty}^{\infty}(-1)^k\tau(n-k(3k-1)/2).$$

$\square$