Problem 11783
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Proposed by Zhang Yun (China).

Given a tetrahedron, let r denote the radius of its inscribed sphere. For $1 \le k \le 4$, let h_k denote the distance from the k-th vertex to the plane of the opposite face. Prove that

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} \ge \frac{12}{5}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The volume of the tetrahedron is given by

$$\frac{h_k A_k}{3} = \frac{rS}{3}$$

where A_k is the area of the face of opposite to the k-th vertex and $S = \sum_{k=1}^{4} A_k$ is the surface area of the tetrahedron. Hence $h_k = r/t_k$ with $t_k = A_k/S \in (0,1)$ and

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} = \sum_{k=1}^{4} f(t_k) \ge 4f\left(\sum_{k=1}^{4} t_k\right) = 4f\left(\frac{1}{4}\right) = \frac{12}{5}$$

where $f(t) = (1-t)/(1+t)$ is a convex function in $[0,+\infty)$. $\square$