Problem 11782
(American Mathematical Monthly, Vol.121, June-July 2014)

Proposed by I. Gessel (USA).

A signed binary representation *of an integer m is a finite list $a_0, a_1, \dots$ of elements of $\{-1, 0, 1\}$ such that $\sum a_i 2^i = m$. A signed binary representation is* sparse *if no two consective entries in the list are nonzero.*
(a) Prove that every integer has a unique sparse representation.
(b) Prove that for all $m \in \mathbb{Z}$, every non-sparse signed binary representation of m has at least as many nonzero terms as the sparse representation.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

(a) We first show that any integer $m \in \mathbb{Z}$ has a sparse signed binary representation.
Consider the ordinary binary representation of $|m|$ and by starting from the right replace any group of $d \geq 2$ consecutive 1s and the next 0 with the equivalent string -1, $d-1$ consecutive 0s and 1:

$$0\underbrace{1\dots1}_{d} \quad \longrightarrow \quad 1\underbrace{0\dots0}_{d-1}-1.$$

Finally, if $m < 0$ then change sign to all digits. At the end of this procedure, all nonzero terms are separated by zeros and we get a sparse representation of m.
As regards uniqueness, if m has two sparse representations then by starting from the left we can compare the corresponding digits and eliminate the ones which are equal. So the two representations are reduced to one of the following cases:

i) $2^r + x = -2^r + y$ where $|x| \leq \lfloor 2^r/3 \rfloor$ and $|y| \leq \lfloor 2^r/3 \rfloor$, which yields a contradiction

$$2^{r+1} = 2^r + 2^r = y - x \leq |x| + |y| \leq 2\lfloor 2^r/3 \rfloor < 2^r.$$

ii) $2^r + x = \pm 2^s + y$ where $r > s$ and $|x| \leq \lfloor 2^r/3 \rfloor$ and $|y| \leq \lfloor 2^s/3 \rfloor$, which yields a contradiction

$$2^r = \pm 2^s + y - x \leq 2^s + |x| + |y| \leq 2^{r-1} + \lfloor 2^r/3 \rfloor + \lfloor 2^{r-1}/3 \rfloor = 2^r - 1 < 2^r.$$

(b) Every signed binary representation can be reduced to the unique sparse representation by starting from the right and by using these replacement rules:

$$\text{i) for } d \geq 2, \quad 0\underbrace{1\dots1}_{d} \longrightarrow 1\underbrace{0\dots0}_{d-1}-1,$$

$$\text{ii) for } d \geq 2, \quad 0\underbrace{-1\cdots-1}_{d} \longrightarrow -1\underbrace{0\dots0}_{d-1}1,$$

$$\text{iii) for } d \geq 1, \quad -1\underbrace{1\dots1}_{d} \longrightarrow \underbrace{0\dots0}_{d}-1,$$

$$\text{iv) for } d \geq 1, \quad 1\underbrace{-1\cdots-1}_{d} \longrightarrow \underbrace{0\dots0}_{d}1.$$

Notice that after any replacement the number of nonzero terms can not increase. Hence every signed binary representation has at least as many nonzero terms as the sparse representation. $\square$