Problem 11780
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Proposed by C. Lupu (USA) and T. Lupu (Romania).

Let f be a positive-valued, concave function on $[0,1]$ *Prove that*

$$\frac{3}{4}\left(\int_0^1 f(x)\,dx\right)^2 \le \frac{1}{8} + \int_0^1 f^3(x)\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We note that, for $t \ge 0$,

$$t^3 - \frac{3}{4}t^2 + \frac{1}{16} = \frac{(4t+1)(2t-1)^2}{16} \ge 0.$$

Hence, since f is non-negative, it follows that

$$\int_0^1 \left(f^3(x) - \frac{3}{4}f^2(x) + \frac{1}{16}\right)\,dx \ge 0,$$

that is, by Cauchy-Schwarz inequality,

$$\int_0^1 f^3(x)\,dx + \frac{1}{16} \ge \frac{3}{4}\int_0^1 f^2(x)\,dx \ge \frac{3}{4}\left(\int_0^1 f(x)\,dx\right)^2$$

which is stronger than the required inequality. The concavity property is not necessary. □