Problem 11778
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Proposed by Li Zhou (USA).

Let x, y, z be positive real numbers such that $x + y + z = \pi/2$. Prove that

$$\sum_{\text{cyc}} \frac{1}{\tan^2 x + 4\tan^2 y + 9\tan^2 z} \leq \frac{9}{14}\left(\tan^2 x + \tan^2 y + \tan^2 z\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $a = \tan^2 x$, $b = \tan^2 y$, and $c = \tan^2 z$, then

$$\sqrt{c} = \tan z = \tan(\pi/2 - (x+y)) = \frac{1 - \tan x \tan y}{\tan x + \tan y} = \frac{1 - \sqrt{ab}}{\sqrt{a} + \sqrt{b}}$$

and therefore $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 1$. Moreover

$$(\sqrt{a} - \sqrt{b})^2 + (\sqrt{b} - \sqrt{c})^2 + (\sqrt{c} - \sqrt{a})^2 \geq 0$$

implies that

$$3(ab + bc + ca) \geq (\sqrt{ab} + \sqrt{bc} + \sqrt{ca})^2 = 1.$$

Hence, it suffices to show that for $a, b, c > 0$,

$$\sum_{\text{cyc}} \frac{1}{a + 4b + 9c} \leq \frac{3(a+b+c)}{14(ab + bc + ca)}.$$

Let $a = \min\{a, b, c\}$, $u = b - a \geq 0$ and $v = c - a \geq 0$, then by expanding the above sum, we find that our inequality is equivalent to

$$(1372a^2 + 108uv)(u^2 - uv + v^2) + 108(u^2 - v^2)^2 + 361uv^3 + u^3v \\ + a(397u(v-u)^2 + 613u^3 + 1010v^3 + 111u^2v) \geq 0$$

which trivially holds. $\square$