Problem 11777
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Proposed by M. Dinca (Romania).

Let $n \geq 3$ *and let* $x_1, \dots, x_n$ *be real numbers such that* $\prod_{k=1}^n x_k = 1$*. Prove that*

$$\sum_{k=1}^n \frac{x_k^2}{x_k^2 - 2x_k \cos(2\pi/n) + 1} \geq 1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Remark: the inequality does not hold for $n = 1$ and $n = 2$ (for example take $x_1 = x_2 = 1$).

If $z_1, \dots, z_n$ and $w_1, \dots, w_n$ are complex numbers then, by the Lagrange's identity,

$$\left(\sum_{k=1}^n |z_k|^2\right)\left(\sum_{k=1}^n |w_k|^2\right) - \left|\sum_{k=1}^n z_k w_k\right|^2 = \sum_{1\leq k<j\leq n} |z_k\overline{w_j} - z_j\overline{w_k}|^2 .$$

Let $w_k = c_k \in \mathbb{R}^+$ and let $z_k = c_k y_k$ with $y_k \in \mathbb{C}$, then the above inequality implies

$$\left(\sum_{k=1}^n c_k^2 |y_k|^2\right)\left(\sum_{k=1}^n c_k^2\right) \geq \sum_{1\leq k<j\leq n} c_k^2 c_j^2 \left|y_k - y_j\right|^2 \geq \sum_{k=1}^n c_k^2 c_{k+1}^2 \left|y_k - y_{k+1}\right|^2$$

where the second inequality holds for $n \geq 3$ with $c_{n+1} = c_1$ and $y_{n+1} = y_1$.
By assuming that $y_1, \dots, y_n$ are distinct and letting $c_k = 1/|y_k - y_{k+1}| > 0$, we obtain

$$\sum_{k=1}^n \frac{|y_k|^2}{|y_k - y_{k+1}|^2} \geq 1.$$

Finally, let $y_{k+1}/y_k = e^{2\pi i/n}/x_k \neq 1$, then $1 = \prod_{k=1}^n (y_{k+1}/y_k) = (e^{2\pi i/n})^n / (\prod_{k=1}^n x_k)$, and we get

$$\sum_{k=1}^n \frac{x_k^2}{x_k^2 - 2x_k \cos(2\pi/n) + 1} = \sum_{k=1}^n \frac{x_k^2}{|x_k - e^{2\pi i/n}|^2} = \sum_{k=1}^n \frac{1}{|1 - y_{k+1}/y_k|^2} = \sum_{k=1}^n \frac{|y_k|^2}{|y_k - y_{k+1}|^2} \geq 1.$$

□