Problem 11775
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Let $A_1, \ldots, A_n$ be finite sets. For $k = 1, \ldots, n$, let $S_k = \sum_{|J|=k} \left|\bigcup_{j\in J} A_j\right|$ with $J \subseteq \{1, \ldots, n\}$.

(a) Express in terms of $S_1, \ldots, S_n$ the number of elements that belong to exactly m of the sets $A_1, \ldots, A_n$.

(b) Same question as in (a), except that we now require the number of elements belonging to at least m of the sets $A_1, \ldots, A_n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

(a) The number of elements that belong to exactly $1 \le m \le n$ of the sets $A_1, \ldots, A_n$ is

$$e_m = (-1)^{n-m} \sum_{j=n-m}^{n} (-1)^{j-1} \binom{j}{n-m} S_j.$$

In fact, an element that is in exactly r of the sets $A_1, \ldots, A_n$ is counted

$$\sum_{i=1}^{r} \binom{r}{i}\binom{n-r}{j-i} = \binom{n}{j} - \binom{n-r}{j}$$

times in S_j where $\binom{r}{i}$ is the number of ways to choose i of the r sets and $\binom{n-r}{j-i}$ is the number of ways complete a set of j elements. Therefore, in the above formula, such an element is counted

$$(-1)^{n-m} \sum_{j=n-m}^{n} (-1)^{j} \binom{j}{n-m} \left(\binom{n-r}{j} - \binom{n}{j}\right) = \begin{cases} 1 & \text{if } r = m, \\ 0 & \text{otherwise,} \end{cases}$$

times because for $0 \le t \le n$,

$$\begin{aligned} \sum_{j=n-m}^{n} (-1)^{j} \binom{j}{n-m}\binom{n-t}{j} &= \binom{n-t}{n-m} \sum_{j=n-m}^{n} (-1)^{j} \binom{m-t}{j-(n-m)} \\ &= (-1)^{n-m}\binom{n-t}{n-m}(1-1)^{m-t} = \begin{cases} (-1)^{n-m} & \text{if } t = m, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

(b) The number of elements belonging to at least $1 \le m \le n$ of the sets $A_1, \ldots, A_n$ is

$$\begin{aligned} a_m &= \sum_{k=m}^{n} e_k = \sum_{k=m}^{n} (-1)^{n-k} \sum_{j=n-k}^{n} (-1)^{j-1} \binom{j}{n-k} S_j = \sum_{j=1}^{n} (-1)^{j-1} S_j \sum_{k=m}^{n} (-1)^{n-k} \binom{j}{n-k} \\ &= \sum_{j=1}^{n} (-1)^{j-1} S_j \sum_{k=0}^{n-m} (-1)^{k} \binom{j}{k} = (-1)^{n-m} \sum_{j=n-m+1}^{n} (-1)^{j-1} \binom{j-1}{n-m} S_j. \end{aligned}$$

□

Note that in a similar way one can prove that

$$e_m = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} T_k \quad \text{and} \quad a_m = \sum_{k=m}^{n} (-1)^{k-m} \binom{k-1}{m} T_k$$

where $T_k = \sum_{|J|=k} \left|\bigcap_{j\in J} A_j\right|$. Moreover for $k = 1, \ldots, n$,

$$S_k = \sum_{j=1}^{k} (-1)^{j-1} \binom{n-j}{k-j} T_j \quad \text{and} \quad T_k = \sum_{j=1}^{k} (-1)^{j-1} \binom{n-j}{k-j} S_j.$$