Problem 11773
(American Mathematical Monthly, Vol.121, April 2014)

Proposed by M. Omarjee (France).

Given a positive real number a_0 , *let* $a_{n+1} = \exp\left(-\sum_{k=0}^{n} a_k\right)$ *for* $n \geq 0$. *For which values of* b *does* $\sum_{n=0}^{\infty}(a_n)^b$ *converge?*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We note that

$$a_{n+1} = \exp\left(-\sum_{k=0}^{n-1} a_k\right) e^{-a_n} = a_n e^{-a_n} = f(a_n)$$

where $f(x) = xe^{-x}$. Since, $f(0) = 0$ and $0 < f(x) < x$ for $x > 0$, it follows that the sequence $\{a_n\}_{n\geq 0}$ is positive and strictly decreasing to 0. Moreover

$$f(x) = x - x^2 + x^3/2 + o(x^3)$$

and by Theorem 2 in *Effective asymptotic for some nonlinear recurrences and almost doubly-exponential sequences*, by Ionascu and Stanica, Acta Mathematica Universitatis Comenianae (2004), we have that

$$a_n = \frac{1}{n} - \frac{\ln(n)}{2n^2} + o\left(\frac{\ln(n)}{n^2}\right).$$

Thus, $\sum_{n=0}^{\infty}(a_n)^b$ converge iff $\sum_{n=1}^{\infty} 1/n^b$ converge, that is for $b > 1$. □