Problem 11772
(American Mathematical Monthly, Vol.121, April 2014)

Proposed by M. Merca (Romania).

Let n be a positive integer. Prove that the number of integer partitions of $2n+1$ that do not contain 1 as a part is less than or equal to the number of integer partitions of $2n$ that contain at least one odd part.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We note that

$$\begin{aligned}
p(2n \mid \text{at least one odd part}) &= p(2n) - p(2n \mid \text{all even parts}) \\
&= p(2n-1) + p(2n \mid \text{no 1-part}) - p(2n \mid \text{all even parts}) \\
&\geq p(2n-1) = p(2n+1 \mid \text{at least one 2-part}) \\
&\geq p(2n+1 \mid \text{no one 1-part})
\end{aligned}$$

where the last inequality is due to the fact that any partition of $2n+1$ with no one 1-part can be transformed to a unique partition of $2n+1$ with at least one 2-part by cutting the smallest part (which is at least 2) into one 2-part and zero or more 1-parts. $\square$