**Problem 11771**
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Proposed by D. M. Batinetu-Giurgiu (Romania) and Neculai Stanciu (Romania).

*Find*

$$\lim_{n\to\infty} \sqrt[n]{(2n-1)!!}\left(\tan\left(\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}\right)-1\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By Stirling's approximation formula

$$\ln(n!) = n\ln(n) - n + \frac{\ln(2\pi n)}{2} + O(1/n),$$

we have that

$$x_n := e\sqrt[n]{n!} = \exp(\ln(n!)/n + 1) = n + \frac{\ln(2\pi n)}{2} + O(1/n)$$

and

$$\lim_{n\to\infty}(x_{n+1}-x_n) = 1 \qquad \text{and} \qquad \lim_{n\to\infty}\frac{x_{an}}{x_n} = a$$

for any positive integer $a$. Now

$$\sqrt[n]{(2n-1)!!} = \sqrt[n]{\frac{(2n)!}{2^n n!}} = \frac{x_{2n}^2}{2x_n} \qquad \text{and} \qquad \lim_{x\to 1}\frac{\tan(\frac{\pi}{4}x)-1}{x-1} = \frac{\pi}{2}.$$

Therefore

$$\sqrt[n]{(2n-1)!!}\left(\tan\left(\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}\right)-1\right) = \frac{x_{2n}^2}{2ex_n}\,\frac{\tan(\frac{\pi}{4}\frac{x_{n+1}}{x_n})-1}{\frac{x_{n+1}}{x_n}-1}\left(\frac{x_{n+1}}{x_n}-1\right)$$
$$= \frac{1}{2e}\left(\frac{x_{2n}}{x_n}\right)^2\frac{\tan(\frac{\pi}{4}\frac{x_{n+1}}{x_n})-1}{\frac{x_{n+1}}{x_n}-1}(x_{n+1}-x_n) \to \frac{\pi}{e}.$$

□