Problem 11770
(American Mathematical Monthly, Vol.121, April 2014)

Proposed by S. P. Andriopoulos (Greece).

Prove, for real numbers a, b, x, y *with* $a > b > 1$ *and* $x > y > 1$, *that*

$$\frac{a^x - b^y}{x - y} > \left(\frac{a+b}{2}\right)^{(x+y)/2} \ln\left(\frac{a+b}{2}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $b < t := (a+b)/2 < a$ then $a^x > t^x$ and $t^y > b^y$. Hence

$$\frac{a^x - b^y}{x - y} > \frac{t^x - t^y}{x - y} > t^{(x+y)/2} \ln(t) = \left(\frac{a+b}{2}\right)^{(x+y)/2} \ln\left(\frac{a+b}{2}\right)$$

where in the second inequality we used the geometric-logarithmic mean inequality

$$\frac{u - v}{\ln(u) - \ln(v)} > \sqrt{uv}$$

for $u = t^x$ and $v = t^y$. $\square$