Problem 11769
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Proposed by P. P. Dályay (Hungary).

Let $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *be positive real numbers. Show that*

$$\left(\sum_{j=1}^{n} \frac{a_j}{b_j}\right)^2 - 2\sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j+b_k)^2} \leq 2\sqrt{2}\left(\sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j+b_k)} \sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j+b_k)^3}\right)^{1/2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $x \in \mathbb{R}$, let $f(x) = \sum_{j=1}^{n} a_j e^{-b_j x}$. Then, for any non-negative integer m,

$$C := \int_0^\infty f(x)\,dx = \sum_{j=1}^{n} \frac{a_j}{b_j}, \qquad D_m := \frac{1}{m!}\int_0^\infty x^m f^2(x)\,dx = \sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j+b_k)^{m+1}},$$

By Cauchy-Schwarz inequality, for $t > 0$,

$$\begin{aligned}\left(\int_0^\infty f(x)dx\right)^2 &= \left(\int_0^\infty \frac{1}{t+x}(t+x)f(x)dx\right)^2 \\ &\leq \left(\int_0^\infty \frac{dx}{(t+x)^2}\right)\left(\int_0^\infty (t+x)^2 f^2(x)dx\right) \\ &= t\int_0^\infty f^2(x)dx + 2\int_0^\infty xf^2(x)dx + \frac{1}{t}\int_0^\infty x^2 f^2(x)dx.\end{aligned}$$

that is

$$C^2 - 2D_1 \leq tD_0 + \frac{2D_2}{t}.$$

By letting $t = \sqrt{2D_2/D_0} > 0$, we obtain $C^2 - 2D_1 \leq 2\sqrt{2}\sqrt{D_2 D_0}$ which is equivalent to the required inequality. □

The original statement of the problem is false. Our correction is based on the paper *A Reverse Hilbert-like Optimal Inequality* by Omran Kouba (arXiv:1404.6744).