Problem 11768
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Proposed by Ovidiu Furdui (Romania).

Let f be a bounded continuous function mapping $[0, +\infty)$ to itself. Find

$$\lim_{n\to\infty} n \left(\sqrt[n]{\int_0^{+\infty} f^{n+1}(x)e^{-x}dx} - \sqrt[n]{\int_0^{+\infty} f^{n}(x)e^{-x}dx} \right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $h(t) = f(-\ln(t))$ then h is a non-negative bounded continuous function in $(0, 1]$.
Let $a_n = \int_0^1 h^n(t)dt$ then

$$L := \lim_{n\to\infty} n \left(\sqrt[n]{\int_0^{+\infty} f^{n+1}(x)e^{-x}dx} - \sqrt[n]{\int_0^{+\infty} f^{n}(x)e^{-x}dx} \right) = \lim_{n\to\infty} n \left(\sqrt[n]{a_{n+1}} - \sqrt[n]{a_n} \right).$$

We will show that $L = M\ln(M)$ where $M := \sup_{x\in[0,+\infty)} f(x) = \sup_{t\in(0,1]} h(t) \geq 0$ (if $M = 0$, then $L = 0$).

If $M = 0$ then f and h are identically zero and the limit is trivial. Let us assume that $M > 0$.
For $0 < \varepsilon < M$, there is a non-empty interval I in $(0, 1]$ such that for all $t \in I$, $h(t) \geq M - \varepsilon > 0$.
Hence

$$(M-\varepsilon)|I|^{1/n} = ((M-\varepsilon)^n|I|)^{1/n} \leq \sqrt[n]{a_n} \leq (M^n|(0,1]|)^{1/n} = M,$$

and, $|I|^{1/n} \to 1$ ($|I| > 0$) implies that $\sqrt[n]{a_n} \to M$.
Now we consider the sequence a_{n+1}/a_n. It is bounded because

$$\frac{a_{n+1}}{a_n} = \frac{1}{a_n}\int_0^1 h^{n+1}(t)dt \leq \frac{M}{a_n}\int_0^1 h^n(t)dt = M.$$

It is increasing because, by Cauchy-Schwarz inequality,

$$a_{n+1}^2 = \left(\int_0^1 h^{\frac{n+2}{2}}(t)h^{\frac{n}{2}}(t)dt\right)^2 \leq \int_0^1 h^{n+2}(t)dt \int_0^1 h^n(t)dt = a_{n+2}\, a_n.$$

So a_{n+1}/a_n has a limit M'. By Stolz-Cesaro theorem, we find that $M' = M$,

$$\ln(M) = \lim_{n\to\infty} \ln(\sqrt[n]{a_n}) = \lim_{n\to\infty} \frac{\ln(a_n)}{n} = \lim_{n\to\infty} (\ln(a_{n+1}) - \ln(a_n)) = \lim_{n\to\infty} \ln\left(\frac{a_{n+1}}{a_n}\right) = \ln(M').$$

Finally,

$$\lim_{n\to\infty} n\left(\sqrt[n]{a_{n+1}} - \sqrt[n]{a_n}\right) = \lim_{n\to\infty} n\sqrt[n]{a_n}\left(\exp\left(\frac{1}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)\right) - 1\right) = \lim_{n\to\infty} \sqrt[n]{a_n}\ln\left(\frac{a_{n+1}}{a_n}\right) = M\ln(M).$$

$\square$