**Problem 11767**
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Proposed by M. Merca (Romania).

*Prove that*

$$\sum \frac{(1+t_1+t_2+\cdots+t_n)!}{(1+t_1)!t_2!\cdots t_n!} = 2^n - F_n$$

*where the sum is over all nonnegative integer solutions to $t_1+2t_2+\cdots+nt_n=n$ and $F_k$ is the kth Fibonacci number.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $F(x,y)=(e^{xy}-1)e^{\frac{x^2y}{1-x}}$ for $x,y\in(-1,1)$ then

$$F(x,y)=(e^{xy}-1)\prod_{k=2}^{\infty}e^{x^2y}=\sum_{t_1=0}^{\infty}\frac{(xy)^{1+t_1}}{(1+t_1)!}\prod_{k=2}^{\infty}\sum_{t_k=0}^{\infty}\frac{(x^ky)^{t_k}}{t_k!}=\sum_{n=0}^{\infty}\sum_{*}\frac{y^{1+t_1+t_2+\cdots+t_n}}{(1+t_1)!t_2!\cdots t_n!}\,x^{n+1}$$

where $\sum_*$ is over all nonnegative integer solutions to $t_1+2t_2+\cdots+nt_n=n$. Hence

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{*}\frac{(1+t_1+t_2+\cdots+t_n)!}{(1+t_1)!t_2!\cdots t_n!}\,x^{n+1} &= \sum_{m=0}^{\infty}\left.\frac{\partial^m(F(x,y))}{\partial y^m}\right|_{y=0}\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\left.\frac{\partial^{m-k}(e^{xy}-1)}{\partial y^{m-k}}\right|_{y=0}\cdot\left.\frac{\partial^k(e^{\frac{x^2y}{1-x}})}{\partial y^k}\right|_{y=0}\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}(x^{m-k}-\delta_{m,k})\cdot\frac{x^{2k}}{(1-x)^k}\\
&=\sum_{m=0}^{\infty}\left(x^m\sum_{k=0}^{m}\binom{m}{k}\frac{x^k}{(1-x)^k}-\frac{x^{2m}}{(1-x)^m}\right)\\
&=\sum_{m=0}^{\infty}\left(x^m\left(1+\frac{x}{1-x}\right)^m-\frac{x^{2m}}{(1-x)^m}\right)\\
&=\sum_{m=0}^{\infty}\frac{x^m-x^{2m}}{(1-x)^m}\\
&=\sum_{m=1}^{\infty}\left(\frac{x}{1-x}\right)^m-\sum_{m=1}^{\infty}\left(\frac{x^2}{1-x}\right)^m\\
&=\frac{x}{1-2x}-\frac{x^2}{1-x-x^2}\\
&=\sum_{n=0}^{\infty}(2^n-F_n)\,x^{n+1},
\end{aligned}$$

which implies that for any nonnegative integer $n$,

$$\sum_{*}\frac{(1+t_1+t_2+\cdots+t_n)!}{(1+t_1)!t_2!\cdots t_n!}=2^n-F_n.$$

□