Problem 11765
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Proposed by D. Beckwith (USA).

Let C_n be the nth Catalan number, given by $C_n = \frac{1}{n+1}\binom{2n}{n}$. Show that

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{C_n} = 5 + \frac{3\pi}{2}$; (b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{C_n} = 22 + 8\sqrt{3}\pi$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that for $|z| < 2$

$$f(z) := (\arcsin(z/2))^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{z^{2n}}{n^2\binom{2n}{n}}.$$

Therefore

$$\begin{aligned} D_z(z^3 D_z(zD_z f)) &= D_z\left(z^3 D_z\left(zD_z \frac{1}{2}\sum_{n=1}^{\infty} \frac{z^{2n}}{n^2\binom{2n}{n}}\right)\right) \\ &= D_z\left(z^3 D_z \sum_{n=1}^{\infty} \frac{z^{2n}}{n\binom{2n}{n}}\right) \\ &= 2D_z \sum_{n=1}^{\infty} \frac{z^{2n+2}}{\binom{2n}{n}} = 4z\sum_{n=1}^{\infty} \frac{z^{2n}}{C_n}. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{C_n} = 1 + \frac{1}{4z} D_z(z^3 D_z(zD_z f)) = g(z) := \frac{2(8+z^2)}{(4-z^2)^2} + \frac{24z\arcsin(z/2)}{(4-z^2)^{5/2}},$$

which implies that

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n}{C_n} = g(\sqrt{2}) = 5 + \frac{3\pi}{2}; \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{3^n}{C_n} = g(\sqrt{3}) = 22 + 8\sqrt{3}\pi.$$

$\square$