Problem 11762
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Proposed by R. P. Stanley (USA).

Let $f(n)$ be the least number of strokes needed to draw the Young diagrams of all the partitions of n. Let

$$F(x) = \sum_{n=1}^{\infty} f(n)x^n = x + 2x^2 + 5x^3 + 12x^4 + 21x^5 + 40x^6 + \cdots .$$

Find the coefficients $g(n)$ of the power series $G(x) = \sum_{n=0}^{\infty} g(n)x^n$ satisfying

$$F(x) = 1 + x + \frac{G(x)}{\prod_{i=1}^{\infty}(1-x^i)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let G be a graph with vertex set V and edge set E then it is known that $2|E| = \sum_{v\in V} \deg(v)$, which implies that the number of vertices of odd degree, is an even number r. We claim that if G is connected then the least number of strokes needed to draw every edge of G is 1 if $r = 0$ and it is $r/2$ if $r > 0$. Indeed, let $v_1, v_2, \ldots, v_{r-1}, v_r$ be the vertices of odd degree, and let us introduce $r/2$ further edges: $\{v_1, v_2\}, \ldots, \{v_{r-1}, v_r\}$. The new graph G' is connected and it has all vertices of even degree. Thus G' has an Eulerian cycle. If one cuts the *new* edges, we get $r/2$ paths covering all the edges of G. Moreover, if we remove from G all the edges of a single path, then the number of vertices of odd degree decreases at most by two (namely the vertices at the start and at the end of the path if it is not a cycle). So if G has $r > 0$ vertices of odd degree, we need at least $r/2$ paths.
Now we consider the connected graph G given by the Young diagram of a partition of $n \geq 1$, say $\sum_{i=1}^{p} a_i n_i = n$ where $1 \leq n_1 < n_2 < \cdots < n_p$ are the different parts of the partition and a_i is the number of copies of the part n_i. Let r be the number of vertices of odd degree of G. Then it is easy to see that $r = 0$ iff $n = 1$ and, for $n \geq 2$, the vertices of odd degree (i. e. 3) are located on the boundary: $\sum_{k=1}^{p} a_k - 1$ on the left vertical line, $n_1 - 1$ on the lower horizontal line, $n_p - 1$ on the upper horizontal line, $\sum_{k=2}^{p}(n_k - n_{k-1} - 1) + \sum_{k=1}^{p}(a_k - 1)$ on the remaining boundary. Hence

$$r = \sum_{k=1}^{p} a_k - 1 + n_1 - 1 + n_p - 1 + \sum_{k=2}^{p}(n_k - n_{k-1} - 1) + \sum_{k=1}^{p}(a_k - 1) = 2\left(\sum_{k=1}^{p} a_k + n_p - p - 1\right).$$

Let $\mathcal{P}_n$ be the set of all partitions of n, then

$$\begin{aligned} F(x) &= x + \sum_{n=2}^{\infty} x^n \sum_{\mathcal{P}_n} \left(\sum_{k=1}^{p} a_k + n_p - p - 1\right) \\ &= x + (F_1(x) - x) + (F_2(x) - x) - (F_3(x) - x) - (P(x) - x - 1) \\ &= 1 + x + F_1(x) + F_2(x) - F_3(x) - P(x) \end{aligned}$$

where F_1 is the g. f. of the total number of parts in all partitions of n, F_2 is the g. f. of the sum of the greater parts in all partitions of n, F_3 is the g. f. of the total number of all different parts in all partitions of n, and P is the g. f. of the number of partitions of n.
By considering the conjugate partition, we see that $F_2 = F_1$. Moreover,

$$F_1(x) = \left[\frac{\partial}{\partial t} \prod_{i=1}^{\infty} \frac{1}{1 - tx^i}\right]_{t=1} = P(x) \sum_{i=1}^{\infty} \frac{x^i}{1 - x^i} \quad \text{and} \quad F_3(x) = \left[\frac{\partial}{\partial t} \prod_{i=1}^{\infty} \left(1 + \frac{tx^i}{1 - x^i}\right)\right]_{t=1} = P(x) \sum_{i=1}^{\infty} x^i.$$

Hence

$$G(x) = \sum_{n=0}^{\infty} g(n)x^n = \frac{F(x) - 1 - x}{P(x)} = \frac{2F_1(x) - F_3(x) - P(x)}{P(x)} = 2\sum_{i=1}^{\infty} \frac{x^i}{1 - x^i} - \sum_{i=1}^{\infty} x^i - 1.$$

Finally $g(n) = 2d(n) - 1$ for $n \geq 0$, where $d(n)$ is the number of divisors of n ($d(0) = 0$).
Note that $F(x) = 1 + x + P(x)G(x)$ yields the following recursive relation: for $n \geq 2$,

$$f(n) = \sum_{k=0}^{n} p(n-k)(2d(k) - 1).$$

□