Problem 11761
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Proposed by B. Tomper (USA).

For each positive integer n, determine the least integer m such that

$$\operatorname{lcm}\{1, 2, \ldots, m\} = \operatorname{lcm}\{n, n+1, \ldots, m\}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $a_1 = 1$, $a_2 = 2$, and, for $n > 2$, let a_n be twice the largest power of a prime less than n.
The first terms of this sequence are

$$1, 2, 4, 6, 8, 10, 10, 14, 16, 18, 18, 22, 22, 26, 26, 26, 32, \ldots.$$

We claim that the least integer m with the required property is just a_n.
Let $n > 2$, and let $a_n = 2p^\alpha$, then $p^\alpha \geq 2$, and, by Bertrand's postulate, there is a prime $q \in (p^\alpha, 2p^\alpha)$. It follows that $2p^\alpha \geq n$, otherwise $p^\alpha < q < 2p^\alpha < n$, contradicting the fact that p^α is largest power of a prime less than n. Moreover, by definition of lcm,

$$\operatorname{lcm}\{1, 2, \ldots, m\} = \prod_{p \text{ prime}} p^{\lfloor \ln(m)/\ln(p) \rfloor} \geq \operatorname{lcm}\{n, n+1, \ldots, m\}$$

and when equality holds for some m_0 then equality is verified also for all $m \geq m_0$.
If $m < a_n = 2p^\alpha$, then $p^\alpha < n \leq m < 2p^\alpha$ implies that

$$p^\alpha \mid \operatorname{lcm}\{1, 2, \ldots, m\}, \quad \text{but} \quad p^\alpha \not| \operatorname{lcm}\{n, n+1, \ldots, m\}$$

so $\operatorname{lcm}\{1, 2, \ldots, m\} > \operatorname{lcm}\{n, n+1, \ldots, m\}$.
It remains to show that if $m = a_n = 2p^\alpha$ then

$$\operatorname{lcm}\{1, 2, \ldots, 2p^\alpha\} = \operatorname{lcm}\{n, n+1, \ldots, 2p^\alpha\}.$$

By the pigeonhole principle, any number in $\{1, 2, \ldots, p^\alpha\}$ has at least one multiple in the set $\{p^\alpha + 1, p^\alpha + 2, \ldots, 2p^\alpha\}$ (which has p^α elements). Hence

$$\operatorname{lcm}\{1, 2, \ldots, 2p^\alpha\} = \operatorname{lcm}\{p^\alpha + 1, p^\alpha + 2, \ldots, 2p^\alpha\}.$$

In order to prove that

$$\operatorname{lcm}\{p^\alpha + 1, p^\alpha + 2, \ldots, 2p^\alpha\} = \operatorname{lcm}\{n, n+1, \ldots, 2p^\alpha\}$$

it suffices to show that if q^β divides $\operatorname{lcm}\{p^\alpha+1, p^\alpha+2, \ldots, n-1\}$ for some prime q, then $q^\beta \leq 2p^\alpha - n + 1$ which implies, by the pigeonhole principle, that there is a multiple of q^β in the set $\{n, n+1, \ldots, 2p^\alpha\}$. It is easy to verify it holds for $n \leq 26$. If $n \geq 26$ then $p^\alpha \geq 25$ and by a stronger version of Bertrand's postulate proved by J. Nagura in *On the interval containing at least one prime number* (1952), there is a prime between p^α and $6p^\alpha/5$. As above, this implies that $6p^\alpha/5 \geq n$. Since p^α is largest power of a prime less than n, it follows that there is an integer $b \geq 2$ such that $bq^\beta \in \{p^\alpha+1, p^\alpha+2, \ldots, n-1\}$. Thus $n > 2q^\beta$ and

$$2p^\alpha - n + 1 > 2p^\alpha - \frac{6p^\alpha}{5} = \frac{4p^\alpha}{5} \geq \frac{4}{5}\,\frac{5n}{6} = \frac{2n}{3} > \frac{n}{2} > q^\beta,$$

and the proof is complete. $\square$