Problem 11759
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Proposed by O. Ganea (Switzerland) and C. Lupu (USA).

Let A be an $n \times n$ skew-symmetric real matrix.
Show that for positive real numbers $x_1, \ldots, x_k$ with $k \geq 2$,

$$\det(A + x_1 I) \cdots \det(A + x_k I) \geq (\det(A + (x_1 \cdots x_k)^{1/k} I))^k.$$

In addition, show that if also all x_i lie on the same side of 1, then

$$\det(A + I)^{k-1} \det(A + (x_1 \cdots x_k) I) \geq \det(A + x_1 I) \cdots \det(A + x_k I).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since A is a $n \times n$ skew-symmetric real matrix, it follows that there exists a $n \times n$ orthogonal matrix Q, and real numbers $\lambda_1, \ldots \lambda_r$, such that

$$A = Q \cdot \operatorname{diag}(B(\lambda_1), \ldots, B(\lambda_r), 0, \cdots, 0) \cdot Q^{-1} \quad \text{where} \quad B(\lambda_i) = \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix},$$

which implies that $\det(A + tI) = t^{n-r} \prod_{j=1}^{r} (t^2 + \lambda_j^2)$. Thus, the first inequality becomes

$$(x_1 \cdots x_k)^{n-r} \prod_{i=1}^{k} \prod_{j=1}^{r} (x_i^2 + \lambda_j^2) \geq \left((x_1 \cdots x_k)^{(n-r)/k} \prod_{j=1}^{r} ((x_1 \cdots x_k)^{2/k} + \lambda_j^2) \right)^k,$$

that is

$$\prod_{j=1}^{r} \left(\prod_{i=1}^{k} (x_i^2 + \lambda_j^2) - ((x_1^2 \cdots x_k^2)^{1/k} + \lambda_j^2)^k \right) \geq 0,$$

which holds because $f_j(t) = \ln(e^t + \lambda_j^2)$ is convex and

$$\ln \left(\prod_{i=1}^{k} (x_i^2 + \lambda_j^2) \right) = \sum_{i=1}^{k} f_j(\ln(x_i^2)) \geq k f_j(\sum_{i=1}^{k} \ln(x_i^2)/k) = \ln \left(((x_1^2 \cdots x_k^2)^{1/k} + \lambda_j^2)^k \right).$$

In a similar way, the second inequality becomes

$$\left(\prod_{j=1}^{r} (1 + \lambda_j^2) \right)^{k-1} \left((x_1 \cdots x_k)^{n-r} \prod_{j=1}^{r} ((x_1 \cdots x_k)^2 + \lambda_j^2) \right) \geq (x_1 \cdots x_k)^{n-r} \prod_{i=1}^{k} \prod_{j=1}^{r} (x_i^2 + \lambda_j^2)$$

that is

$$\prod_{j=1}^{r} \left(\frac{x_1^2 \cdots x_k^2 + \lambda_j^2}{1 + \lambda_j^2} - \prod_{i=1}^{k} \frac{x_i^2 + \lambda_j^2}{1 + \lambda_j^2} \right) \geq 0$$

which holds by the following argument. The function $g_j(t) = f_j(t) - \ln(1 + \lambda_j^2)$ is convex, increasing and $g_j(0) = 0$ which implies that it is superadditive in $[0, +\infty)$ and in $(-\infty, 0]$.
Hence, if all x_i lie on the same side of 1 then all $\ln(x_i^2)$ lie on the same side of 0, and

$$\ln \left(\frac{x_1^2 \cdots x_k^2 + \lambda_j^2}{1 + \lambda_j^2} \right) = g_j \left(\sum_{i=1}^{k} \ln(x_i^2) \right) \geq \sum_{i=1}^{k} g_j(\ln(x_i^2)) = \ln \left(\prod_{i=1}^{k} \frac{x_i^2 + \lambda_j^2}{1 + \lambda_j^2} \right).$$

□